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Analytic treatment of the two loop equal mass sunrise graph

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Abstract

The two loop equal mass sunrise graph is considered in the continuous d-dimensional regularisation for arbitrary values of the momentum transfer. After recalling the equivalence of the expansions at $d = 2$ and $d = 4$, the second order differential equation for the scalar Master Integral is expanded in $(d-2)$ and solved by the variation of the constants method of Euler up to first order in $(d-2)$ included. That requires the knowledge of the two independent solutions of the associated homogeneous equation, which are found to be related to the complete elliptic integrals of the first kind of suitable arguments. The behaviour and expansions of all the solutions at all the singular points of the equation are exhaustively discussed and written down explicitly.

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1 Introduction.

The two-loop sunrise selfmass graph with equal masses is not yet known analytically, despite its apparent simplicity. In this paper we fill that gap and evaluate it, for arbitrary values of the momentum transfer, in terms of a suitable set of new functions, whose analytic properties (in particular behaviours and expansions at all their potentially singular points) are worked out explicitly in full details.

We follow the differential equation approach, already introduced in [1] in the arbitrary mass case for studying particular values of the momentum transfer, within the usual regularization scheme in d-continuous dimensions. We are interested, as usual, in the $d \to 4$ limit; but it is known from [2] how to relate algebraically the coefficients of the expansions at $d = 4$ and those of the expansion at $d = 2$, and as the expansion at $d = 2$ turns out to be simpler, we discuss in fact the $d \to 2$ limit.

In the equal mass limit the two loop sunrise has two Master Integrals (MI's) only, which satisfy a system of two first order differential equations in the momentum transfer u (we will use often also the Euclidean variable $z = -u$, which is positive when u is spacelike), equivalent to a single second order equation for the simpler of the two MI's. The equation, which is exact in d, when expanded around $d = 2$ in powers of $(d-2)$ gives rise to a set of chained inhomogeneous equations all having the same homogeneous part; we solve them recursively by the variation of the constants method of Euler, which gives formally the solutions of the inhomogeneous equations in terms of the solutions of the common homogeneous equation. Only when the homogeneous solutions are known Euler's formula does provide an effective analytic evaluation; the bulk of the paper consists, indeed, in working out those homogeneous solutions.

The paper is organized as follow. In Section 2 the differential equations are written; in Section 3 the connection between the expansions at $d = 4$ and at $d = 2$ is established; in Section 4 the expansion of the second order differential equation at $d = 2$ is worked out. Section 5 discusses how the solutions of the homogeneous equation can be obtained; Section 6 presents the expansions in z of the homogeneous solutions at all the singular points of the differential equation; Section 7 discusses another set of functions, represented in the form of one-dimensional definite integral over a suitable parameter, which solve the homogeneous equation between any two nearby singular points (referred to as "interpolating solutions"). Section 8 then finally gives the proper analytic continuation of the homogeneous solutions for any value of z, and Section 9 discusses the properties of the homogeneous equation and of its solutions for special conformal transformations of the argument.

Once the homogeneous solutions are known, Section 10 works out explicitly the MI as given by Euler's formula for the solution of the inhomogeneous equation at zeroth order in $(d-2)$, Section 11 the corresponding zeroth order term of the expansion in $(d-4)$ and Section 12 the first order term of the expansion in $(d-2)$.

Section 13 contains the conclusions; the first of the two Appendices deals with the relation between the "interpolating solutions" and the complete elliptic integrals of the first kind, the second Appendix with some definite integrals occurring in Sections 10 and 12.

2 The differential equations for the MIs.

The 2-loop sunrise graph of Fig. 1 with arbitrary masses m_1, m_2, m_3, is known to possess 4 Master Integrals (MI's) [3], already extensively studied in the literature [4]. The system of the four linear differential equations in the squared external momentum p^2 for the Master Integrals (MIs) was written in [1]. The differential equations were used for obtaining analytically specific values or behaviours at zero and infinite momentum transfer [1], at pseudothresholds [5] and threshold [6], as well as for direct numerical integration [7], but a satisfactory knowledge of the analytic expression of the MI's for arbitrary momentum transfer and masses is still missing. In this paper we tackle that problem in the simpler equal masses case. While the algebraic burden in the arbitrary mass case will surely be much heavier, there are however indications [8] that the approach can be extended to the arbitrary mass case as well.

We will work as usual within the d-continuous regularization scheme, defining the loop integration

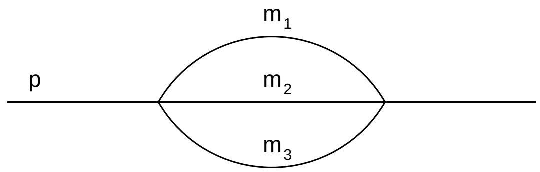

Figure 1: The 2-loop sunrise graph.

measure, in agreement with previous work, as

$$\mathfrak{D}^d k = \frac{1}{C(d)} \frac{d^d k}{(2\pi)^{d-2}} \ ,$$

with $C(d) = (4\pi)^{(4-d)/2}\Gamma(3-d/2)$, *i.e.*

$$\mathfrak{D}^d k = \frac{1}{\Gamma\left(3-\frac{d}{2}\right)} \frac{d^d k}{4\pi^{\frac{d}{2}}} \ . \tag{2.1}$$

At $d = 4$, it reduces to the standard measure

$$\mathfrak{D}^4 k = \frac{d^4 k}{(2\pi)^2} \ ; \tag{2.2}$$

further, with the definition Eq.(2.1), the tadpole reads (exactly in d)

$$T(d,m) = \int \frac{\mathfrak{D}^d k}{k^2+m^2} = \frac{m^{d-2}}{(d-2)(d-4)} \ . \tag{2.3}$$

When all the masses are equal, the 2-loop sunrise has just two MIs; putting equal to 1 the value of the mass, the two MI's $S(d,p^2)$ and $S_1(d,p^2)$ can be defined as

$$S(d,p^2) = \int \frac{\mathfrak{D}^d k_1 \ \mathfrak{D}^d k_2}{(k_1^2+1)(k_2^2+1)[(p-k_1-k_2)^2+1]} \ , \tag{2.4}$$

$$S_1(d,p^2) = \int \frac{\mathfrak{D}^d k_1 \mathfrak{D}^d k_2}{(k_1^2+1)^2(k_2^2+1)[(p-k_1-k_2)^2+1]} \ , \tag{2.5}$$

where p is the external momentum. In the following we will use

$$p^2 = z = -u \ , \tag{2.6}$$

with z positive (and u negative) when p is spacelike, and the continuation to timelike values (u positive) will be performed with the usual replacement

$$z = -(u + i\epsilon) \ . \tag{2.7}$$

The system of linear differential equations in z satisfied by the 4 MI's in the arbitrary mass case was written in [1]; in the equal mass limit it goes into the following 2×2 system of linear differential equations

in z for the two MI's defined in Eq.s(2.4,2.5)

$$
\begin{aligned}
z\frac{d}{dz}S(d,z) &= (d-3)S(d,z)+3S_1(d,z)\ , \\
z(z+1)(z+9)\frac{d}{dz}S_1(d,z) &= \frac{1}{2}(d-3)(8-3d)(z+3)S(d,z) \\
&+ \frac{1}{2}\left[(d-4)z^2+10(2-d)z+9(8-3d)\right]S_1(d,z) \\
&+ \frac{1}{2}\frac{z}{(d-4)^2}\ .
\end{aligned} \tag{2.8}
$$

The system can be rewritten as a second order differential equation for $S(d,z)$ only

$$
\begin{aligned}
& z(z+1)(z+9)\frac{d^2}{dz^2}S(d,z) \\
&+ \frac{1}{2}\left[(12-3d)z^2+10(6-d)z+9d\right]\frac{d}{dz}S(d,z) \\
&+ \frac{1}{2}(d-3)\left[(d-4)z-d-4\right]S(d,z) = \frac{3}{2}\frac{1}{(d-4)^2}\ ,
\end{aligned} \tag{2.9}
$$

and the second MI $S_1(d,z)$ can be expressed in terms of $S(d,z)$ and its first derivative,

$$
S_1(d,z) = \frac{1}{3}\left[-(d-3)+z\frac{d}{dz}\right]S(d,z)\ . \tag{2.10}
$$

From now on we can take $S(d,z)$ and its first derivative $dS(d,z)/dz$ as the effective MIs of the problem.

3 The expansions in d.

In any physical application, the $d \to 4$ limit is to be taken; in that limit, $S(d,z)$ can be Laurent-expanded in $(d-4)$ around $d=4$ as

$$
S(d,z) = \sum_{n=-2} S^{(n)}(4,z)(d-4)^n\ , \tag{3.1}
$$

where, as it is well known, the Laurent-series at $d=4$ starts with a leading double pole in $(d-4)$. In this paper we will show how to evaluate analytically the coefficients $S^{(n)}(4,z)$ up to $n=1$, *i.e.* up to the first order term in $(d-4)$ included.

Let us recall here that by acting on any Feynman integral in d-continuous dimensions with a suitable differential operator one can obtain an expression relating the value of that integral evaluated in $(d-2)$ dimensions to the values, evaluated in d dimensions, of a suitable combination of other integrals related to the same Feynman graph [2]. In the case of the Master Integral $S(d,z)$, expressing the result in terms of the effective MI's (and of tadpoles) the relation reads

$$
\begin{aligned}
S(d-2,z) = \frac{4}{(d-6)^2(z+1)(z+9)}\Bigg[& (d-3)\big(3(d-2)+2(d-3)z\big)S(d,z) \\
& - 2(d-3)z(z-3)\frac{d}{dz}S(d,z) + \frac{3(z+3)}{4(d-4)^2}\Bigg]\ .
\end{aligned} \tag{3.2}
$$

By differentiating in z the above relation, and using Eq(2.9) for eliminating the second derivative of $S(d,z)$, one obtains a similar relation for $dS(d-2,z)/dz$. The two relations can be used to express $S(d,z)$ in terms of $S(d-2,z)$ and $dS(d-2,z)/dz$; replacing further d by $(2+d)$, one finally obtains

$$
\begin{aligned}
S(2+d,z) = \frac{1}{12(d-1)(3d-2)(3d-4)}\Bigg\{ & 2(d-4)^2(z+1)(z+9)\left[1+(z-3)\frac{d}{dz}\right]S(d,z) \\
& +(d-2)(d-4)^2(87+22z-z^2)S(d,z) - \frac{36}{(d-2)^2} + \frac{3z-63}{(d-2)}\Bigg\}
\end{aligned} \tag{3.3}
$$

Quite in general, assume a relation of the form

$$L(2+d) = R(d) \tag{3.4}$$

as given; setting $d = 2+\eta$ and Laurent-expanding in η around $d = 2$ both the l.h.s., $L(4+\eta) = \sum_n L^{(n)}(4)\eta^n$, and the r.h.s., $R(2+\eta) = \sum_n R^{(n)}(2)\eta^n$, one obtains

$$\sum_n L^{(n)}(4)\eta^n = \sum_n R^{(n)}(2)\eta^n \ ,$$

which obviously implies, at any order n in the Laurent-expansion in η

$$L^{(n)}(4) = R^{(n)}(2).$$

We put $d = 2 + \eta$ in Eq.(3.3) and look for the systematical Laurent-expansion of l.h.s. and r.h.s. The l.h.s. is nothing but

$$\sum_n S^{(n)}(4,z)\eta^n \ ,$$

where the coefficients $S^{(n)}(4,z)$ are the same as in Eq.(3.1). Within the r.h.s. of Eq.(3.3) we write

$$S(d,z) = \sum_{n=0} S^{(n)}(2,z)\eta^n \ , \tag{3.5}$$

where the sum starts from $n = 0$ as $S(d,z)$, Eq.(2.4), is regular at $d = 2$; the singularities in $(d-2)$ of the r.h.s. are then entirely due to the two last terms in the r.h.s. of Eq.(3.3), with the double and simple poles in $(d-2)$ coming from the tadpoles entering in Eq.(3.2). That implies in particular that the Laurent-expansion of the l.h.s. must also have a double and a simple pole, as already anticipated. When expanding also the overall d-depending coefficient, one finds

$$\begin{aligned}
S^{(-2)}(4,z) &= -\frac{3}{8} \ , &(3.6)\\
S^{(-1)}(4,z) &= \frac{1}{32}(z+18) \ , &(3.7)\\
S^{(0)}(4,z) &= \frac{1}{12}(z+1)(z+9)\left(1+(z-3)\frac{d}{dz}\right)S^{(0)}(2,z) &\\
&\quad -\frac{1}{128}(72+13z) \ . &(3.8)\\
S^{(1)}(4,z) &= \frac{1}{12}(z+1)(z+9)\left[1+(z-3)\frac{d}{dz}\right]S^{(1)}(2,z) &\\
&\quad + \frac{1}{48}(21-126z-19z^2)S^{(0)}(2,z) &\\
&\quad - \frac{17}{48}(z+1)(z+9)(z-3)\frac{dS^{(0)}(2,z)}{dz} &\\
&\quad + \frac{5}{512}(36+23z) \ . \ . &(3.9)
\end{aligned}$$

Note that the singular part of $S(d,z)$ for $d \to 4$ is entirely determined by the above equations (and of course in agreement with previous results, see for instance Eq.s(53) of [1]).

In the following of this paper we will show how to obtain $S^{(0)}(2,z)$ and $S^{(1)}(2,z)$; $S^{(0)}(4,z)$ and $S^{(1)}(4,z)$ can then be obtained by the previous formulae.

4 The expansion around $d = 2$.

It was shown in the previous Section, Eq.(3.6-3.9) that the coefficients $S^{(n)}(4,z)$ of the expansion of $S(d,z)$ around $d = 4$ can be expressed in terms of the coefficients $S^{(n)}(2,z)$ of the expansion around $d = 2$, which

is perhaps also simpler, as polar terms in $(d-2)$ are absent. In the following we will therefore restrict ourselves to the evaluation of the coefficients $S^{(n)}(2,z)$ of the expansion at $d=2$, referring to Eq.s(3.6-3.9) for obtaining the $S^{(n)}(4,z)$.

By inserting the expansion Eq.(3.5) into Eq.(2.9) and systematically expanding in $(d-2)$ also all the other d-depending terms, one obtains a system of chained equations, each equation corresponding to a given order in $(d-2)$. All the equations are of the form

$$\left\{ \frac{d^2}{dz^2} + \left[\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right]\frac{d}{dz} + \left[\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)}\right]\right\} S^{(n)}(2,z) = N^{(n)}(2,z) \ , \tag{4.1}$$

where $n = 0, 1, ..$ is the order in the expansion in $(d-2)$. Note that the differential operator acting on $S^{(n)}(2,z)$ is the same for any value of n, while the inhomogeneous terms do depend on n. The explicit expressions of the first two terms are

$$\begin{aligned} N^{(0)}(2,z) &= \frac{1}{24z} - \frac{3}{64(z+1)} + \frac{1}{192(z+9)} = \frac{3}{8z(z+1)(z+9)} \ , \\ N^{(1)}(2,z) &= \left(-\frac{1}{2z} + \frac{1}{z+1} + \frac{1}{z+9}\right)\frac{dS^{(0)}(2,z)}{dz} \\ &+ \left(\frac{5}{18z} - \frac{1}{8(z+1)} - \frac{11}{72(z+9)}\right) S^{(0)}(2,z) \\ &+ \frac{1}{24z} - \frac{3}{64(z+1)} + \frac{1}{192(z+9)} \ . \end{aligned} \tag{4.2}$$

The system of equations Eq.(4.1,4.2) is chained, in the sense that the equation for the coefficient $S^{(n)}(2,z)$ involves in general in the inhomogeneous term coefficients $S^{(k)}(2,z)$ (and their first derivatives) of lower order $k<n$. Therefore the system must to be solved bottom up; as $N^{(0)}(2,z)$ is explicitly known, one starts from $n=0$ and obtains $S^{(0)}(2,z)$, which then appears as a known term in the inhomogeneous part of the equation at $n=1$ for $S^{(1)}(2,z)$, and so on to higher values of n.

The homogeneous part of Eq.(4.1), which as already observed is the same for any value of n, reads

$$\left\{ \frac{d^2}{dz^2} + \left[\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right]\frac{d}{dz} + \left[\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)}\right]\right\} \Psi(z) = 0 \ . \tag{4.3}$$

As it is a second order differential equation, it admits two linearly independent solutions, say $\Psi_1,(z), \Psi_2,(z)$; if

$$W(z) = \Psi_1(z)\frac{d\Psi_2(z)}{dz} - \Psi_2(z)\frac{d\Psi_1(z)}{dz} \ , \tag{4.4}$$

is their Wronskian, according to Euler's method of the variation of the constants, the solutions of Eq.(4.1) are given by the integral representations

$$\begin{aligned} S^{(n)}(2,z) &= \Psi_1(z)\left(\Psi_1^{(n)} - \int_0^z \frac{dw}{W(w)}\Psi_2(w)N^{(n)}(2,w)\right) \\ &+ \Psi_2(z)\left(\Psi_2^{(n)} + \int_0^z \frac{dw}{W(w)}\Psi_1(w)N^{(n)}(2,w)\right) \ , \end{aligned} \tag{4.5}$$

where $\Psi_1^{(n)}, \Psi_2^{(n)}$ are two integration constants.

Up to this point Eq.(4.5) is just formal: it will become a "substancial", explicit analytic expression only when all the ingredients entering in it – the two $\Psi_i(z)$, their Wronskian and the integration constants $\Psi_i^{(k)}$ – will be actually evaluated.

Although the Wronskian is defined in terms of the $\Psi_i(z)$, it is known (Liouville's formula) that it can be immediately obtained (up to a multiplicative constant) from Eq.(4.3). An elementary calculation using

the definition Eq.(4.4) and the value of the second derivatives of the $\Psi_i(z)$, as given by Eq.(4.3) of which they are solutions, leads to the equation

$$\frac{d}{dz}W(z) = -\left(\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right) W(z) \ ,$$

which gives at once

$$W(z) = \frac{9}{z(z+1)(z+9)} \ , \tag{4.6}$$

where the multiplicative constant has been fixed anticipating later results.

Evaluating the two $\Psi_i(z)$ (and then the integration constants) is much more laborious.

5 Building the solution of the homogeneous equation.

Given a differential equation, it is immediate to obtain its solutions around any point z_0 in the form of a power series in $(z - z_0)$; indeed, when the solution is tentatively written as a power series, the differential equation can be easily recast as a recursive formula for the coefficients of the expansion. If the differential equation is of second order, as in Eq.(4.3), the specification of the two initial conditions which identify the solution corresponds to the specification of the first two coefficients, needed for initializing the recursion.

The expansion converges only up to the nearest singular point; in the case of Eq.(4.3) the singular points are the four points $z = 0, -1, -9$ and $z = \infty$. At each of the four singular points, one finds (see below) that Eq.(4.3) has a regular solution, say $\psi_1(z)$, whose series expansion is straightforward, and a logarithmically divergent solution; the divergent solution can be written as the logarithm of the appropriate argument times the same function $\psi_1(z)$ which corresponds to the regular solution, plus another regular function, say $\psi_2(z)$, whose series expansion coefficients are also recursively provided by the equation.

While obtaining a local solution is immediate, building a solution in the whole range $-\infty < z < \infty$ (*i.e.* working out the analytic continuation of a local solution) is a much more demanding task. At least in principle, one could consider unrelated pairs of linearly independent solutions evaluated as series expansions around some carefully chosen points, say $\bar{z}_j$, each solution depending on a pair of constants (unrelated, to begin with), pick up one of the points, say $\bar{z}_0$, and start defining the two independent solutions at that point by fixing somehow the two constants at that point. One can then look at the expansion around the nearest point, say $\bar{z}_1$ (it is assumed that thanks to the careful choice of the points $\bar{z}_j$ the convergence regions of the expansions in $\bar{z}_0$ and $\bar{z}_1$ overlap) and fix the two arbitrary constants of the expansion at $\bar{z}_1$ (*i.e.* obtain the analytic continuation from $\bar{z}_0$ to $\bar{z}_1$) by imposing the equality of the solutions in the overlapping region. The procedure, which can be implemented without significant loss of precision also in a merely numerical way, can then be iterated to cover the whole range of interest of z.

In order to cover with properly converging expansions also the singular points, which are at the boundary of the convergence regions of the expansions around nearby points, it is mandatory to consider also the expansions around the singular points themselves. Therefore, we will first work out the (still unrelated) solutions around each of the four singular points as series expansions; for joining them into a unique, analytically continued pair of solutions, rather than relying on expansions around auxiliary points (as in a numerical approach) we will look for "interpolating solutions", valid within the intervals between successive pairs of singular points – namely the three intervals $(0 > z > -1)$, $(-1 > z > -9)$ and $(-9 > z > -\infty)$ – and then will use the limiting values of the interpolating solution at the boundaries of each interval for relating the arbitrary constants of the solutions at the singular points, building in such a way the desired analytic continuation of the solutions in the whole range of the variable z.

6 The solutions at the singular points.

The presence of $1/z$ factors in the coefficients of Eq.(4.3) shows that $z = 0$ is a singular point of the equation; when looking for a solution whose leading power in z is z^α one finds the indicial equation $\alpha^2 = 0$,

indicating that the leading behaviors at $z = 0$ are 1 and $\log z$, so that around $z = 0$ the two independent solutions can be written as

$$\begin{aligned} \Psi_1^{(0)}(z) &= \psi_1^{(0)}(z) , \\ \Psi_2^{(0)}(z) &= \ln z \; \psi_1^{(0)}(z) + \psi_2^{(0)}(z) . \end{aligned} \tag{6.1}$$

When $\psi_1^{(0)}(z), \psi_2^{(0)}(z)$ are expanded as power series in z as

$$\psi_i^{(0)}(z) = \sum_{n=0}^{\infty} \psi_{i,n}^{(0)} \; z^n \; ; \qquad\qquad i = 1, 2 \tag{6.2}$$

Eq.(4.3) gives for the coefficients $\psi_i^{(0)}(n)$ the recursion relations valid for $n \geq 1$

$$\begin{aligned} \psi_{1,n}^{(0)} = \frac{1}{9n^2} \quad & \Big[(-3 + 10n - 10n^2) \psi_{1,n-1}^{(0)} \\ & -(n-1)^2 \psi_{1,n-2}^{(0)} \Big] \; , \\ \psi_{2,n}^{(0)} = \frac{1}{9n^3} \quad & \Big[(-10n + 6) \psi_{1,n-1}^{(0)} \\ & -2(n-1) \psi_{1,n-2}^{(0)} \\ & -n(10n^2 - 10n + 3) \psi_{2,n-1}^{(0)} \\ & -n(n-1)^2 \psi_{2,n-2}^{(0)} \Big] \; . \end{aligned} \tag{6.3}$$

The initial conditions

$$\begin{aligned} \psi_{i,n}^{(0)} &= 0 \quad \text{if} \quad n < 0 \\ \psi_{1,0}^{(0)} &= 1 \\ \psi_{2,0}^{(0)} &= 0 \; , \end{aligned} \tag{6.4}$$

determine uniquely the coefficients of the expansions. For definiteness, the first terms are

$$\begin{aligned} \psi_1^{(0)}(z) &= 1 - \frac{1}{3} z + \frac{5}{27} z^2 + ... \\ \psi_2^{(0)}(z) &= -\frac{4}{9} z + \frac{26}{81} z^2 + ... \end{aligned} \tag{6.5}$$

The radius of convergence of the series Eq.(6.2) is 1, as the nearby singularity of the solutions of Eq.(4.3) is $z = -1$ or $u = 1$, see Eq.(2.7). We were unable to solve the recurrence relations Eq.(6.3) in closed form, but by direct inspection one easily see that at large n the coefficients have a $1/n$ behaviour (which in turn implies that the radius of convergence is 1). For $0 < z$, *i.e.* $0 > u$ (the Euclidean region), both solutions $\Psi_1^{(0)}(z), \Psi_2^{(0)}(z)$, Eq.(6.1) are real; as for spacelike u there are no singular points of Eq.(4.3) in the whole range $0 > u > -\infty$, both functions can be continued along the whole positive z-axis $0 < z < +\infty$, where they keep real values. For $-1 < z < 0$, if $z = -(u + i\epsilon)$, with $1 > u > 0$ and $\epsilon > 0$, $\Psi_1^{(0)}(z)$ is still real, while

$$\ln z = \ln(-u - i\epsilon) = \ln u - i\pi \; ,$$

so that $\Psi_2^{(0)}(-u - i\epsilon)$ develops an imaginary part equal to $-i\pi\psi_1^{(0)}(-u)$.

Around $z = -1$, one finds similarly that two independent solutions of Eq.(4.3) are given by

$$\begin{aligned} \Psi_1^{(1)}(z) &= \psi_1^{(1)}(z) , \\ \Psi_2^{(1)}(z) &= \ln(z+1) \; \psi_1^{(1)}(z) + \psi_2^{(1)}(z) , \end{aligned} \tag{6.6}$$

where $\psi_1^{(1)}(z), \psi_2^{(1)}(z)$ are defined by the series expansions

$$\psi_i^{(1)}(z) = \sum_{n=0}^{\infty} \psi_{i,n}^{(1)} (z+1)^n \ , \qquad\qquad i = 1,2 \tag{6.7}$$

with initial conditions

$$\begin{aligned} \psi_{i,n}^{(1)} &= 0 \quad \text{if} \quad n < 0 \\ \psi_{1,0}^{(1)} &= 1 \\ \psi_{2,0}^{(1)} &= 0 \ . \end{aligned} \tag{6.8}$$

The recursion relations valid for $n \geq 1$ are

$$\begin{aligned} \psi_{1,n}^{(1)} = \frac{1}{8n^2} & \Big[(7n^2 - 7n + 2)\psi_{1,n-1}^{(1)} \\ & + (n-1)^2 \psi_{1,n-2}^{(1)} \Big] \ , \\ \psi_{2,n}^{(1)} = \frac{1}{8n^3} & \Big[(7n-4)\psi_{1,n-1}^{(1)} \\ & + 2(n-1)\psi_{1,n-2}^{(1)} \\ & + n(7n^2 - 7n + 2)\psi_{2,n-1}^{(1)} \\ & + n(n-1)^2 \psi_{2,n-2}^{(1)} \Big] \ . \end{aligned} \tag{6.9}$$

For definiteness, the first terms of the expansions are

$$\begin{aligned} \psi_1^{(1)}(z) &= 1 + \frac{1}{4}(z+1) + \frac{5}{32}(z+1)^2 + ... \\ \psi_2^{(1)}(z) &= \frac{3}{8}(z+1) + \frac{33}{128}(z+1)^2 + ... \end{aligned} \tag{6.10}$$

As in the previous case, it was not possible to solve Eq.s(6.9) in closed form, but the coefficients are easily seen to behave as $1/n$ for large n, so that the radius of convergence of the series Eq.(6.7) is 1, the nearest singularity of the solutions of Eq.(4.3) being at $z = 0$. Similarly to the previous case, $\Psi_1^{(1)}(z)$ is real in the whole range of convergence of the expansion around $z = -1$, while $\Psi_1^{(2)}(z)$ is real for $z > -1$ and develops an imaginary part $-i\pi\psi_1^{(1)}(z)$ for $z = -(u + i\epsilon)$, $u > 1$.

Around $z = -9$ one has similarly

$$\begin{aligned} \Psi_1^{(9)}(z) &= \psi_1^{(9)}(z) \ , \\ \Psi_2^{(9)}(z) &= \ln(z+9)\ \psi_1^{(9)}(z) + \psi_2^{(9)}(z) \ , \end{aligned} \tag{6.11}$$

where $\psi_1^{(9)}(z), \psi_2^{(9)}(z)$ are given by the series expansions

$$\psi_i^{(9)}(z) = \sum_{n=0}^{\infty} \psi_{i,n}^{(9)}\ (z+9)^n \ , \qquad\qquad i = 1,2 \tag{6.12}$$

with the coefficients $\psi_{i,n}^{(9)}$ determined by the initial conditions

$$\begin{aligned} \psi_{i,n}^{(9)} &= 0 \quad \text{if} \quad n < 0 \\ \psi_{1,0}^{(9)} &= 1 \\ \psi_{2,0}^{(9)} &= 0 \ , \end{aligned} \tag{6.13}$$

and by the recursion relations valid for $n \geq 1$

$$
\begin{aligned}
\psi^{(9)}_{1,n} = \frac{1}{72n^2} \quad & \Big[(17n^2 - 17n + 6)\psi^{(9)}_{1,n-1} \\
& -(n-1)^2\psi^{(9)}_{1,n-2}\Big] \ , \\
\psi^{(9)}_{2,n} = \frac{1}{72n^3} \quad & \Big[(17n - 12)\psi^{(9)}_{1,n-1} \\
& -2(n-1)\psi^{(9)}_{1,n-2} \\
& +n(17n^2 - 17n + 6)\psi^{(9)}_{2,n-1} \\
& -n(n-1)^2\psi^{(9)}_{2,n-2}\Big] \ .
\end{aligned} \tag{6.14}
$$

For definiteness, the first terms of the expansions are

$$
\begin{aligned}
\psi^{(9)}_1(z) &= 1 + \frac{1}{12}(z+9) + \frac{7}{864}(z+9)^2 + ... \\
\psi^{(9)}_2(z) &= \frac{5}{72}(z+9) + \frac{97}{10368}(z+9)^2 + ...
\end{aligned} \tag{6.15}
$$

The expected radius of convergence of the series is 8 (the nearest singularity is at $z = -1$ or $u = 1$), $\Psi^{(9)}_1(z)$ being real on the whole interval of convergence, while $\Psi^{(9)}_2(z)$ develops an imaginary part $-i\pi\psi^{(9)}_1(z)$ for $z = -(u + i\epsilon)$, $u > 9$.

Finally, for $z \to \infty$ we put $z = 1/y$ and find as independent solutions

$$
\begin{aligned}
\Psi^{(\infty)}_1(z) &= y\psi^{(\infty)}_1(y) \ , \\
\Psi^{(\infty)}_2(z) &= y\left(\ln(y)\ \psi^{(\infty)}_1(y) + \psi^{(\infty)}_2(y)\right) \ ,
\end{aligned} \tag{6.16}
$$

where $\psi^{(\infty)}_1(y), \psi^{(\infty)}_2(y)$ are the following two series in y

$$
\psi^{(\infty)}_i(y) = \sum_{n=0}^{\infty} \psi^{(\infty)}_{i,n} y^n \ , \qquad i = 1,2 \tag{6.17}
$$

with the coefficients $\psi^{(\infty)}_{i,n}$ determined by the initial conditions

$$
\begin{aligned}
\psi^{(\infty)}_{i,n} &= 0 \quad \text{if} \quad n < 0 \\
\psi^{(\infty)}_{1,0} &= 1 \\
\psi^{(\infty)}_{2,0} &= 0 \ ,
\end{aligned} \tag{6.18}
$$

and by the recursion relations valid for $n \geq 1$

$$
\begin{aligned}
\psi^{(\infty)}_{1,n} = -\frac{1}{n^2} \quad & \Big[(10n^2 - 10n + 3)\psi^{(\infty)}_{1,n-1} \\
& +9(n-1)^2\psi^{(\infty)}_{1,n-2}\Big] \ , \\
\psi^{(\infty)}_{2,n} = -\frac{1}{n^2} \quad & \Big[(10 - 6/n)\psi^{(\infty)}_{1,n-1} \\
& +18(1 - 1/n)\psi^{(\infty)}_{1,n-2} \\
& +(10n^2 - 10n + 3)\psi^{(\infty)}_{2,n-1} \\
& +9(n-1)^2\psi^{(\infty)}_{2,n-2}\Big] \ .
\end{aligned} \tag{6.19}
$$

For definiteness, the first terms of the expansions are

$$\begin{aligned} \psi_1^{(\infty)}(z) &= 1 - 3y + 15y^2 + ... \\ \psi_2^{(\infty)}(z) &= -4y + 26y^2 + ... \end{aligned} \tag{6.20}$$

The expected radius of convergence of the series in $y = 1/z$ is $1/9$, as the nearest singularity is at $u = -z = 9$. For $y > 0$, *i.e.* $z > 0$, the two series are real; for $z = -(u + i\epsilon)$, $\infty > u > 9$, $\ln y = -\ln u + i\pi$ and $\Psi_2^{(\infty)}(z)$ develops an imaginary part $-i\pi/u\ \psi_1^{(\infty)}(y)$.

7 The interpolating solutions.

Eq.s(4.1),(4.2) at $n = 0$ give the equation valid for the $n = 2$ limiting value of $S(d, z)$, Eq.(2.4); as the inhomogeneous term $N^{(0)}(2, z)$, Eq.(4.2), is always real for real z, the imaginary part of $S^{(0)}(2, z)$ satisfies the homogeneous equation Eq.(4.3). As the imaginary part is present only above the physical threshold, *i.e.* when $u = -z > 9$, it is convenient to rewrite Eq.(4.3) in terms the function $J(u) = \Psi(z)$ and of the variable $u = -z$,

$$\left\{ \frac{d^2}{du^2} + \left[\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-9}\right] \frac{d}{du} + \left[-\frac{1}{3u} + \frac{1}{4(u-1)} + \frac{1}{12(u-9)}\right]\right\} J(u) = 0 \ . \tag{7.1}$$

As already observed in in [9], the Cutkosky-Veltman rule gives for the imaginary part of $S^{(0)}(2, z)$ (up to a multiplicative constant irrelevant here), the integral representation

$$\int_4^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{R_4(u,b)}} \ , \tag{7.2}$$

where $R_4(u, b)$ stands for the polynomial (of 4th order in b and 2nd order in u)

$$R_4(u, b) = b(b-4)(b-(\sqrt{u}-1)^2)(b-(\sqrt{u}+1)^2) \ . \tag{7.3}$$

For $u > 9$ the four roots in b of $R_4(u, b)$ are ordered as

$$0 < 4 < (\sqrt{u}-1)^2 < (\sqrt{u}+1)^2 \ , \tag{7.4}$$

and the b integration in Eq.(7.2) runs between the two adjacent roots 4 and $(\sqrt{u}-1)^2$. We will now verify that Eq.(7.2) does indeed satisfy Eq.(7.1) by a procedure which will also suggest a broader set of similar solutions, also for other ranges of values of u.

To that aim, let us introduce the auxiliary functions

$$H(k, u) = \int_\beta^B \frac{db\ b^k}{\sqrt{R_4(u,b)}} \ , \tag{7.5}$$

β, B are any pair of adjacent roots in b of $R_4(u, b)$, and k is an integer (if the integration boundaries do not include the root $b = 0$, k can take any positive or negative value; otherwise, $k \geq 0$ for simplicity). One finds immediately

$$\int_\beta^B db \frac{d}{db}\left(b^n \sqrt{R_4(u,b)}\right) = 0 \ ,$$

as the integration runs among two points at which the primitive of the integrand vanish; by working out explicitly the derivatives, moving always the square root $\sqrt{R_4(u,b)}$ to the denominator and using the definition Eq.(7.5) one finds the identity

$$\begin{aligned} (k+2)H(k+3,u) - (2k+3)(u+3)H(k+2,u) + (k+1)(u+3)^2 H(k+1,u) & \\ -2(2k+1)(u-1)^2 H(k,u) &= 0 \ . \end{aligned} \tag{7.6}$$

Another identity is obtained by writing

$$\int_\beta^B db \frac{d}{db}\left(\ln\frac{b(u-b+3)+\sqrt{R_4(u,b)}}{b(u-b+3)-\sqrt{R_4(u,b)}}\right) = 0 \ ;$$

the derivative of the logarithm is

$$\frac{-3b+(u+3)}{\sqrt{R_4(u,b)}} \ ,$$

from which one finds

$$3H(1,u)-(u+3)H(0,u) = 0 \ . \tag{7.7}$$

As a consequence of the identities Eq.s(7.6,7.7), all the integrals $H(k,u)$, Eq.(7.5) can be expressed in terms of only two "master integrals", which can be taken to be

$$\begin{aligned} H(0,u) &= \int_\beta^B \frac{db}{\sqrt{R_4(u,b)}} \ , \\ H(2,u) &= \int_\beta^B \frac{db\ b^2}{\sqrt{R_4(u,b)}} \ . \end{aligned} \tag{7.8}$$

Also the u-derivatives of all the $H(k,u)$ can be expressed in terms of the two functions. By using Eq.s(7.6,7.7) write

$$\begin{aligned} \int_\beta^B db\sqrt{R_4(u,b)} &= \left(\frac{3}{2}+\frac{21}{2}u-\frac{3}{2}u^2+\frac{1}{6}u^3\right)H(0,u)-\left(\frac{3}{2}+\frac{1}{2}u\right)H(2,u) \\ \int_\beta^B dbb\ \sqrt{R_4(u,b)} &= \left(\frac{15}{4}+\frac{61}{4}u+\frac{49}{6}u^2+\frac{7}{6}u^3+\frac{1}{12}u^4+\frac{1}{36}u^5\right)H(0,u) \\ &\quad +\left(-\frac{15}{4}+\frac{3}{4}u-\frac{9}{4}u^2-\frac{1}{12}u^3\right)H(2,u) \ . \end{aligned}$$

One can then take the u-derivatives of both sides of the two equations; in the l.h.s. one can differentiate the integrands, and then express the result in terms of the two functions $H(0,u)$, $H(2,u)$ times polynomials in u; in the r.h.s. one finds both the two functions and their u-derivatives; solving for the derivatives gives

$$\begin{aligned} \frac{d}{du}H(0,u) &= \frac{1}{2u(u-1)(u-9)}\left[(3+14u-u^2)H(0,u)-3H(2,u)\right] \\ \frac{d}{du}H(2,u) &= \frac{1}{6u(u-1)(u-9)}\left[(9+228u+30u^2-12u^3+u^4)H(0,u)\right. \\ &\quad \left. +(-9-42u+3u^2)H(2,u)\right] \ . \end{aligned} \tag{7.9}$$

With one further differentiation and some algebra one obtains also the second derivative

$$\begin{aligned} \frac{d^2}{du^2}H(0,u) &= \frac{1}{2u^2(u-1)^2(u-9)^2}\left[(-27-12u+202u^2-36u^3+u^4)H(0,u)\right. \\ &\quad \left. +(27-60u+9u^2)H(2,u)\right] \ . \end{aligned} \tag{7.10}$$

One can now substitute $J(u)$ by $H(0,u)$ in Eq.(7.1); by using Eq.s(7.9,7.10) for expressing the derivatives in terms of $H(0,u)$ and $H(2,u)$, one finds that Eq.(7.1) is satisfied. But $H(0,u)$ Eq.(7.8) is equal to the integral Eq.(7.2) if $\beta=4$ and $B=(\sqrt{u}-1)^2$; it is so verified that Eq.(7.2) satisfy Eq.(7.8), as expected.

The verification that $H(0,u)$ satisfies Eq.(7.8) remains valid for any choice of the two roots (β, B) out of the four roots of $R_4(u,b)$ listed in Eq.(7.4). The linearly independent choices can be taken to be the three real integrals

$$J_1(u) = \int_0^4 \frac{db}{\sqrt{-R_4(u,b)}} \ , \qquad J_2(u) = \int_4^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{R_4(u,b)}} \ , \qquad J_3(u) = \int_{(\sqrt{u}-1)^2}^{(\sqrt{u}+1)^2} \frac{db}{\sqrt{-R_4(u,b)}} \ , \tag{7.11}$$

where the sign in front of $R_4(u,b)$ within the square root has been adjusted to make all the integrals real (but an overall multiplicative factor i would be anyhow irrelevant when dealing with the solutions of a homogeneous equation). All the three functions $J_i(u), i = 1,2,3$ are solutions of Eq.(4.3), which has however only two linearly independent solutions; therefore they cannot be all independent. Indeed, consider the contour integral

$$\oint_{(\infty)} db \, \frac{1}{\sqrt{R_4(u,b)}} \ , \tag{7.12}$$

where the loop runs on the circle at infinity. For $u > 9$ the integrand has two cuts; one cut is from $b = (\sqrt{u}-1)^2$ to $b = (\sqrt{u}+1)^2$ and correspondingly its value is $1/\sqrt{R_4(u,b)} = 1/(i\sqrt{-R_4(u,b)})$ for b above the cut and $-1/(i\sqrt{-R_4(u,b)})$ for b below the cut, the other cut from $b = 0$ to $b = 4$, and corresponding values $-1/(i\sqrt{-R_4(u,b)})$ and $1/(i\sqrt{-R_4(u,b)})$ above and below the cut. The integral on the circle at infinity (where the integrand behaves as $1/b^2$) vanishes; by shrinking the circle to two closed paths around the cuts, one obtains, apart an overall factor $2i$,

$$J_1(u) - J_3(u) = 0 \ , \tag{7.13}$$

which reduces by one, as expected, the number of the independent solutions.

The ordering Eq.(7.4) of the four roots in b of $R_4(u,b)$ is valid, as already remarked, for $u > 9$; when u varies, the ordering of the roots varies as well, and the corresponding real interpolating solutions of the form of Eq.(7.11) are expressed by different choices of the integration interval in the variable b.

8 The homogeneous solutions.

We will now use the results of the previous Sections for building two independent solutions of Eq.(4.3), say $\Psi_1(z), \Psi_2(z)$, properly continued in the whole range $-\infty < z < \infty$. We start from the the point $z = 0$; in the interval $1 > z > -1$ we define

$$1 > z > -1 \quad \begin{cases} \Psi_1(z - i\epsilon) = \Psi_1^{(0)}(z - i\epsilon) \ , \\ \\ \Psi_2(z - i\epsilon) = \Psi_2^{(0)}(z - i\epsilon) \ , \end{cases} \tag{8.1}$$

where the two functions $\Psi_1^{(0)}(z), \Psi_2^{(0)}(z)$ are defined by Eq.(6.1) and the equations following it. With that definition, the multiplicative constant of the Wronskian is fixed, and the Wronskian takes the value

$$W(z) = \frac{9}{z(z+1)(z+9)} \ ,$$

already anticipated in Eq.(4.6).

Eq.(8.1) is meaningful within the convergence region of the series, *i.e.* $1 > z > -1$. For positive z the $-i\epsilon$ prescription is irrelevant, as the two functions $\Psi_i(z)$ are both real; for $0 > z > -1$, i.e. $0 < u < 1$, $\Psi_1(z)$ is still real, while the factor $\ln(z - i\epsilon)$ present in $\Psi_2(z)$ develops an imaginary part $-i\pi$, so that one has

$$0 > z > -1 \quad \begin{cases} \Psi_1(z - i\epsilon) = \psi_1^{(0)}(z) \ , \\ \\ \Psi_2(z - i\epsilon) = \ln(-z)\psi_1^{(0)}(z) + \psi_2^{(0)}(z) - i\pi\psi_1^{(0)}(z) \ , \end{cases} \tag{8.2}$$

and, quite in general, for real z

$$\Psi_i(z+i\epsilon) = (\Psi_i(z-i\epsilon))^* \ . \tag{8.3}$$

In the $z \to 0^-$ limit, in particular, one finds

$$\begin{aligned} \lim_{z\to 0^-} \Psi_1(z-i\epsilon) &= 1 \ , \\ \lim_{z\to 0^-} \Psi_2(z-i\epsilon) &= \ln(-z) - i\pi \ . \end{aligned} \tag{8.4}$$

In the interval $0 > z > -1$, *i.e.* or $0 < u < 1$, the four roots of $R_4(u,b)$, Eq.(7.3), are ordered as

$$(0, (\sqrt{u}-1)^2, (\sqrt{u}+1)^2, 4) \ ;$$

according to Section 7, the interpolating solutions can be chosen as

$$\begin{aligned} J_1^{(0,1)}(u) &= \int_0^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{-R_4(u,b)}} \ , \\ J_2^{(0,1)}(u) &= \int_{(\sqrt{u}-1)^2}^{(\sqrt{u}+1)^2} \frac{db}{\sqrt{R_4(u,b)}} \ , \\ J_3^{(0,1)}(u) &= \int_{(\sqrt{u}+1)^2}^{4} \frac{db}{\sqrt{-R_4(u,b)}} \ . \end{aligned} \tag{8.5}$$

The above functions are real, and regular in the region $0 < u < 1$; according to the discussion leading to Eq.(7.13), one has however the identity

$$J_3^{(0,1)}(u) = J_1^{(0,1)}(u) \ , \tag{8.6}$$

showing that only two of them can be linearly independent. In the $z \to 0^-$ or $u \to 0^+$ limit, the two roots $(\sqrt{u}-1)^2$ and $(\sqrt{u}+1)^2$ become equal, so that the integration in b of Eq.s(8.5) is elementary and gives

$$\begin{aligned} \lim_{u\to 0^+} J_1^{(0,1)}(u) &= \frac{1}{\sqrt{3}}\left(-\frac{1}{2}\ln u + \ln 3\right) \ , \\ \lim_{u\to 0^+} J_2^{(0,1)}(u) &= \frac{\pi}{\sqrt{3}} \ . \end{aligned} \tag{8.7}$$

The explicit calculation gives also

$$\lim_{u\to 0^+} J_3^{(0,1)}(u) = \frac{1}{\sqrt{3}}\left(-\frac{1}{2}\ln u + \ln 3\right)$$

in agreement, of course, with Eq.s(8.6) and (8.7).

Given the different behaviours in the $u \to 0^+$ limit, $J_1^{(0,1)}(u)$ and $J_2^{(0,1)}(u)$ are linearly independent; in that same limit, one finds that their Wronskian

$$\lim_{u\to 0^+}\left[J_1^{(0,1)}(u)\frac{d}{du}J_2^{(0,1)}(u) - \frac{d}{du}J_1^{(0,1)}(u)J_2^{(0,1)}(u)\right] = \pi\frac{1}{6} \ ,$$

shows the expected $1/u$ singularity (the $J_i^{(0,1)}(u)$ are solutions of Eq.(7.1), which is equivalent to Eq.(4.3) with $z = -u$ and whose Wronskian Eq.(4.6) behaves as $1/z$ when $z \to 0$, up to an overall multiplicative factor).

The $\Psi_i(z), i = 1, 2,$ are linearly independent solutions of Eq.(4.3), so that the $J_i^{(0,1)}(u), i = 1, 2,$ which are also solutions of the same equation (the equivalence of Eq.(7.1) and Eq.(4.3) has been repeatedly

recalled) can be expressed in terms of the $\Psi_i(z)$, and *viceversa*. By comparing the $z \to 0^-$ behaviour of the $\Psi_i(z)$ given by Eq.s(8.4) and the $u \to 0^+$ behaviour of the $J_i^{(0,1)}(u)$, Eq.s(8.7), one finds that in the interval $0 > z > -1$, *i.e.* $0 < u < 1$ the solutions can be written as

$$
\begin{aligned}
u = -z, \qquad & 0 > z > -1, \quad 0 < u < 1, \\
\Psi_1(z - i\epsilon) &= \frac{\sqrt{3}}{\pi} J_2^{(0,1)}(u) \, , \\
\Psi_2(z - i\epsilon) &= -2\sqrt{3} J_1^{(0,1)}(u) + \frac{\sqrt{3}}{\pi} \left(2\ln 3 - i\pi \right) J_2^{(0,1)}(u) \, .
\end{aligned} \tag{8.8}
$$

Similarly, in the $u \to 1^-$ limit, the two couples of roots $(0, (\sqrt{u}-1)^2)$ and $((\sqrt{u}+1)^2, 4)$ become equal, so that one finds easily

$$
\begin{aligned}
\lim_{u \to 1-} J_1^{(0,1)}(u) &= \lim_{u \to 1-} J_3^{(0,1)}(u) = \frac{\pi}{4} \, , \\
\lim_{u \to 1-} J_2^{(0,1)}(u) &= -\frac{3}{4} \ln(1-u) + \frac{9}{4} \ln 2 \, .
\end{aligned} \tag{8.9}
$$

But for $u \to 1^-$ (or $z \to -1^+$) the two interpolating solutions $J_i^{(0,1)}(u)$ can also be expressed in terms of the two $\Psi_i^{(1)}(z)$, Eq.s(6.6); matching the limiting values of Eq.s(8.9) with the corresponding behaviours of Eq.s(6.6) (the $\Psi_i^{(1)}(z)$ are real for $0 < z < -1$) gives

$$
0 > z > -1 \quad \begin{cases} J_1^{(0,1)}(-z) = \frac{\pi}{4} \Psi_1^{(1)}(z) \, , \\[1ex] J_2^{(0,1)}(-z) = \frac{9}{4} \ln 2 \; \Psi_1^{(1)}(z) - \frac{3}{4} \Psi_2^{(1)}(z) \, . \end{cases} \tag{8.10}
$$

Substituting Eq.s(8.10) into Eq.s(8.8), we finally obtain

$$
0 > z > -2 \quad \begin{cases} \Psi_1(z - i\epsilon) = \frac{9\sqrt{3}}{4\pi} \ln 2 \Psi_1^{(1)}(z - i\epsilon) - \frac{3\sqrt{3}}{4\pi} \Psi_2^{(1)}(z - i\epsilon) \, , \\[1ex] \Psi_2(z - i\epsilon) = \frac{\sqrt{3}}{4} \left(\frac{18}{\pi} \ln 2 \ln 3 - 2\pi - i 9 \ln 2 \right) \Psi_1^{(1)}(z - i\epsilon) \\ \qquad + \frac{3\sqrt{3}}{4\pi} \left(-2\ln 3 + i\pi \right) \Psi_2^{(1)}(z - i\epsilon) \, , \end{cases} \tag{8.11}
$$

which gives the required analytic continuation of the solutions $\Psi_i(z)$, as defined by Eq.(8.1), from the interval $1 > z > -1$ to the interval $0 > z > -2$ (or $0 < u < 2$), containing the singular point $z = -1$ (or $u = 1$).

The whole procedure can be repeated in the interval $-9 > z > -1$ or $1 < u < 9$. The four roots of $R_4(u, b)$, Eq.(7.3), are ordered as

$$
(0, \; (\sqrt{u}-1)^2, \; 4, \; (\sqrt{u}+1)^2) \, ,
$$

and the two interpolating solutions are given by

$$
\begin{aligned}
J_1^{(1,9)}(u) &= \int_0^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{-R_4(u,b)}} \, , \\
J_2^{(1,9)}(u) &= \int_{(\sqrt{u}-1)^2}^{4} \frac{db}{\sqrt{R_4(u,b)}} \, ,
\end{aligned} \tag{8.12}
$$

while for the third solution, according to Eq.(7.13), one has

$$
J_3^{(1,9)}(u) = \int_4^{(\sqrt{u}+1)^2} \frac{db}{\sqrt{-R_4(u,b)}} = J_1^{(1,9)}(u) \, .
$$

In the $u \to 1^+$ limit (as in the $u \to 1^-$ limit) the pairs of roots $(0, (\sqrt{u}-1)^2)$ and $((\sqrt{u}+1)^2, 4)$ become equal and one evaluates easily

$$\begin{aligned} \lim_{u\to 1^+} J_1^{(1,9)}(u) &= \frac{\pi}{4} \ , \\ \lim_{u\to 1^+} J_2^{(1,9)}(u) &= -\frac{3}{4}\ln(u-1) + \frac{9}{4}\ln 2 \ . \end{aligned} \tag{8.13}$$

As the $J_i^{(1,9)}(u)$ are solutions of Eq.(7.1), equivalent to Eq.(4.3), in the interval $1 < u < 9$ or $-1 > z > -9$, they can be written as linear combinations of the $\Psi_i(z - i\epsilon)$; the comparison of Eq.s(8.13) with Eq.s(6.6,6.10) in the $z \to -1^-$ limit gives

$$-1 > z > -9 \quad \begin{cases} \Psi_1(z-i\epsilon) = \frac{\sqrt{3}}{\pi}\left(J_2^{(1,9)}(u) + 3iJ_1^{(1,9)}(u)\right) \ , \\ \\ \Psi_2(z-i\epsilon) = \frac{\sqrt{3}}{\pi}\Big((\pi + 6i\ln 3)J_1^{(1,9)}(u) \\ \qquad\qquad +(2\ln 3 - i\pi)J_2^{(1,9)}(u)\Big) \ . \end{cases} \tag{8.14}$$

We look now at $u \to 9^-$; in that limit, $(\sqrt{u}-1)^2 \to 4$, and one obtains

$$\begin{aligned} \lim_{u\to 9^-} J_1^{(1,9)}(u) &= \frac{1}{4\sqrt{3}}\left(-\ln(9-u) + 3\ln 2 + 2\ln 3\right) \ , \\ \lim_{u\to 9^-} J_2^{(1,9)}(u) &= \frac{\pi}{4\sqrt{3}} \ . \end{aligned} \tag{8.15}$$

In close analogy to what already done for the interval $0 < u < 1$, in the interval $1 < u < 9$ the two interpolating solutions $J_i^{(1,9)}(u)$ can be expressed in terms of the $\Psi_i^{(9)}(z)$ Eq.s(6.11); matching the coefficients of the linear combination by using the limiting values for $u \to 9^-$, one obtains

$$-1 > z > -9 \quad \begin{cases} J_1^{(1,9)}(u) = \frac{1}{4\sqrt{3}}\left((3\ln 2 + 2\ln 3)\Psi_1^{(9)}(z) - \Psi_2^{(9)}(z)\right), \\ \\ J_2^{(1,9)}(u) = \frac{\pi}{4\sqrt{3}}\Psi_1^{(9)}(z) \ . \end{cases} \tag{8.16}$$

When substituting Eq.s(8.16) in Eq.s(8.14) one finally obtains

$$-1 > z > -17 \quad \begin{cases} \Psi_1(z) = \frac{1}{4\pi}\left[\pi + i\ 3(3\ln 2 + 2\ln 3)\right]\Psi_1^{(9)}(z) \\ \qquad\quad - i\ \frac{3}{4\pi}\Psi_2^{(9)}(z) \ , \\ \\ \Psi_2(z) = \left[\frac{3}{4}\ln 2 + \ln 3 + \frac{i}{4\pi}(18\ln 2\ln 3 + 12\ln^2 3 - \pi^2)\right]\Psi_1^{(9)}(z) \\ \qquad\quad - \frac{1}{4\pi}(\pi + i\ 6\ln 3)\Psi_2^{(9)}(z) \ . \end{cases} \tag{8.17}$$

The radius of convergence of the expansion around $z = -9$, *i.e.* $u = 9$, of the $\Psi_i^{(9)}(z)$ is 8, as the nearest singularity is at $u = 1$ (or $z = -1$), hence the range of validity $-1 > z > -17$ or $1 < u < 17$ of the above formula.

The last interval to consider is $9 < u < \infty$; the four roots of $R_4(u, b)$ are ordered as

$$0, 4, (\sqrt{u}-1)^2, (\sqrt{u}+1)^2 \quad ,$$

and two interpolating solutions corresponding to integrating in b between two adjacent roots are

$$\begin{aligned} J_1^{(9,\infty)}(u) &= \int_0^4 \frac{db}{\sqrt{-R_4(u,b)}} \ , \\ J_2^{(9,\infty)}(u) &= \int_4^{(\sqrt{u}-1)^2} \frac{db}{\sqrt{R_4(u,b)}} \ , \end{aligned} \tag{8.18}$$

while the third solution satisfies

$$J_3^{(9,\infty)}(u) = \int\limits_{(\sqrt{u}-1)^2}^{(\sqrt{u}+1)^2} \frac{db}{\sqrt{-R_4(u,b)}} = J_1^{(9,\infty)}(u) \ .$$

Proceedings as int the previous case, from the limiting values for $u \to 9^+$

$$\begin{aligned} \lim_{u\to 9^+} J_1^{(9,\infty)}(u) &= \frac{1}{4\sqrt{3}}\Big(-\ln(u-9) + 3\ln 2 + 2\ln 3\Big) \ , \\ \lim_{u\to 9^+} J_2^{(9,\infty)}(u) &= \frac{\pi}{4\sqrt{3}} \ . \end{aligned} \tag{8.19}$$

one obtains

$$-9 > z > -\infty \quad \begin{cases} \Psi_1(z-i\epsilon) = \frac{\sqrt{3}}{\pi}\left(-2J_2^{(9,\infty)}(u) + 3iJ_1^{(9,\infty)}(u)\right) \ , \\ \\ \Psi_2(z-i\epsilon) = \frac{\sqrt{3}}{\pi}\Big((\pi + 6i\ln 3)J_1^{(9,\infty)}(u) \\ \qquad\qquad -4\ln 3 J_2^{(9,\infty)}(u)\Big) \ . \end{cases} \tag{8.20}$$

The limiting values for $u \to \infty$ are

$$\begin{aligned} \lim_{u\to\infty} J_1^{(9,\infty)}(u) &= \frac{1}{u}\pi \ , \\ \lim_{u\to\infty} J_2^{(9,\infty)}(u) &= \frac{3}{2u}\ln u \ , \end{aligned} \tag{8.21}$$

implying

$$-9 > z > -\infty \quad \begin{cases} J_1^{(9,\infty)}(u) = -\pi\Psi_1^{(\infty)}(z-i\epsilon) \ , \\ \\ J_2^{(9,\infty)}(u) = -\frac{3}{2}i\pi\Psi_1^{(\infty)}(z-i\epsilon) + \frac{3}{2}\Psi_2^{(\infty)}(z-i\epsilon) \ . \end{cases} \tag{8.22}$$

Note that according to the definitions of the $\Psi_i^{(\infty)}(z)$, Eq.(6.16), the r.h.s. of the above equations are real, as expected (according to the definition Eq.(8.18), the $J_i^{(9,\infty)}(u)$ are indeed real).

By substituting Eq.s(8.22) into Eq.s(8.20) one finally obtains,

$$-9 > z > -\infty \quad \begin{cases} \Psi_1(z-i\epsilon) = -3\frac{\sqrt{3}}{\pi}\Psi_2^{(\infty)}(z-i\epsilon) \ , \\ \\ \Psi_2(z-i\epsilon) = -\sqrt{3}\pi\Psi_1^{(\infty)}(z-i\epsilon) - 6\sqrt{3}\frac{\ln 3}{\pi}\Psi_2^{(\infty)}(z-i\epsilon) \ . \end{cases} \tag{8.23}$$

The formula holds also in the range $\infty > z > 9$ (where the $i\epsilon$ in the arguments can be ignored).

9 Identities under transformations of the argument.

Consider the transformation

$$y = \frac{9}{z} \ , \qquad z = \frac{9}{y} \ . \tag{9.1}$$

It maps the singular points of Eq.(4.3), $z = 0, -1, -9, \infty$ into themselves, more exactly $0 \to \infty \to 0$, $(-1) \to (-9) \to (-1)$. If z is real and negative, y is also real and negative, and the $i\epsilon$ prescription is implemented as

$$\frac{9}{z-i\epsilon} = \frac{9}{z} + i\epsilon = y + i\epsilon \ , \qquad \frac{9}{y+i\epsilon} = \frac{9}{y} - i\epsilon = z - i\epsilon \ . \tag{9.2}$$

As it is easy to verify, if $\Psi(z)$ is a solution of Eq.(4.3),

$$\frac{1}{z}\Psi\left(\frac{9}{z}\right) \tag{9.3}$$

is also a solution of the same equation, and can be therefore expressed in terms of the solutions $\Psi_1(z)$ and $\Psi_2(z)$ defined in the previous Section. The argument applies, of course, to the $\Psi_i(z)$ themselves, for which we can therefore write

$$\Psi_i\left(\frac{9}{z} - i\epsilon\right) = \frac{1}{3}\ z \sum_{j=1,2} A_{ij}\Psi_j(z+i\epsilon)\ , \tag{9.4}$$

where the coefficients A_{ij} are the elements of a 2×2 matrix A and the overall factor $1/3$ has been introduced for convenience. One can obtain the four coefficients A_{ij} by imposing for instance Eq.(9.4) to be valid for $z = -1+\eta, y = -9-\eta$, with $\eta \to 0^+$, were the limiting values of the $\Psi_i(z)$ are known from Eq.s(6.6,6.11). The result is

$$A = \frac{\sqrt{3}}{\pi}\begin{pmatrix} 2\ln 3 & -1 \\ 4\ln^2 3 - \frac{1}{3}\pi^2 & -2\ln 3 \end{pmatrix}\ . \tag{9.5}$$

One can check that with the above values of the A_{ij} Eq.(9.4) holds also in the $z \to 0$ limit, when the limiting values of the $\Psi_i(z)$ are given by Eq.s(6.1,6.16).

Eq.(9.4) can also be written, letting $z \to 9/z$, as

$$\Psi_i(z-i\epsilon) = 3\frac{1}{z}\sum_{j=1,2} A_{ij}\Psi_j\left(\frac{9}{z}+i\epsilon\right)\ ;$$

as the matrix A is real, by taking the complex conjugate and recalling Eq.(8.3) one has

$$\Psi_i(z+i\epsilon) = 3\ \frac{1}{z}\sum_{j=1,2} A_{ij}\Psi_j\left(\frac{9}{z}-i\epsilon\right)\ , \tag{9.6}$$

When chaining Eq.(9.6) and Eq.(9.4) one finds the condition

$$A\cdot A = 1\ , \tag{9.7}$$

which is of course satisfied by the explicit values given in Eq.(9.5). Eq.(9.7) shows also that two of the four coefficients A_{ij} are fixed once the other two are given.

The transformation Eq.(9.1) maps the points $z = -3, 3$ into themselves. Correspondingly, Eq.s(9.4,9.5) give

$$\begin{aligned} \Psi_2(3) &= \frac{1}{\sqrt{3}}(2\sqrt{3}\ln 3 - \pi)\ \Psi_1(3)\ , \\ Re\Psi_2(-3) &= \frac{1}{\sqrt{3}}(2\sqrt{3}\ln 3 + \pi)\ Re\Psi_1(-3)\ , \\ Im\Psi_2(-3) &= \frac{1}{\sqrt{3}}(2\sqrt{3}\ln 3 - \pi)\ Im\Psi_1(-3)\ . \end{aligned} \tag{9.8}$$

In terms of the interpolating solutions, Eq.s(9.4) become: in the interval $0 < u < 1$, with $\infty > 9/u > 9$,

$$\begin{aligned} 0 < \quad & u \quad < 1 \\ J_1^{(9,\infty)}\left(\frac{9}{u}\right) &= u\frac{\sqrt{3}}{9}J_2^{(0,1)}(u)\ , \\ J_2^{(9,\infty)}\left(\frac{9}{u}\right) &= u\frac{\sqrt{3}}{3}J_1^{(0,1)}(u)\ ; \end{aligned} \tag{9.9}$$

in the interval $1 < u < 9$, with $9 > 9/u > 1$,

$$\begin{aligned} 1 < \quad & u \quad < 9 \\ J_1^{(1,9)}\left(\frac{9}{u}\right) &= u\frac{\sqrt{3}}{9}J_2^{(1,9)}(u)\ , \\ J_2^{(1,9)}\left(\frac{9}{u}\right) &= u\frac{\sqrt{3}}{3}J_1^{(1,9)}(u)\ , \end{aligned} \tag{9.10}$$

and, in the interval $9 < u < \infty$, with $1 > 9/u > 0$,

$$
\begin{aligned}
9 < \quad & u \quad < \infty \\
J_1^{(0,1)}\left(\frac{9}{u}\right) &= u\frac{\sqrt{3}}{9}J_2^{(9,\infty)}(u) \ , \\
J_2^{(0,1)}\left(\frac{9}{u}\right) &= u\frac{\sqrt{3}}{3}J_1^{(9,\infty)}(u) \ .
\end{aligned} \tag{9.11}
$$

Eq.s(9.11) are of course equivalent to Eq.s(9.9); similarly, the second of Eq.s(9.10) is equivalent to the first. At $u = 3$, in particular, they reduce to

$$
J_2^{(1,9)}(3) = \sqrt{3}J_1^{(1,9)}(3) \ . \tag{9.12}
$$

We consider next the transformation

$$
y = -\frac{z+9}{z+1} \ , \qquad z = -\frac{y+9}{y+1} \ . \tag{9.13}
$$

which also maps into themselves the singular points of Eq.(4.3), more exactly it maps $0 \to (-9) \to 0,$ and $(-1) \to \infty \to (-1)$. For real y, z the transformation implies for the $i\epsilon$ prescription

$$
\begin{aligned}
-\frac{z - i\epsilon + 9}{z - i\epsilon + 1} &= -\frac{z+9}{z+1} - i\epsilon = y - i\epsilon \ , \\
-\frac{y - i\epsilon + 9}{y - i\epsilon + 1} &= -\frac{y+9}{y+1} - i\epsilon = z - i\epsilon \ .
\end{aligned} \tag{9.14}
$$

One can verify that, if $\Psi(z)$ is a solution of Eq.(4.3),

$$
\frac{1}{z+1}\Psi\left(-\frac{z+9}{z+1}\right) \tag{9.15}
$$

is also a solution of the same equation, and therefore a combination of $\Psi_1(z)$ and $\Psi_2(z)$. When applied to the $\Psi_i(z)$ themselves the argument gives

$$
\Psi_i\left(-\frac{z+9}{z+1} - i\epsilon\right) = \frac{\sqrt{2}}{4}\ (z+1)\sum_{j=1,2} B_{ij}\Psi_j(z - i\epsilon) \ , \tag{9.16}
$$

where the coefficients B_{ij} are the elements of a 2×2 matrix B and the overall factor $\sqrt{2}/4$ has been introduced for convenience. The B_{ij} can be fixed, for instance, by imposing the validity of Eq.(9.16) for $z \to 0$; the result reads

$$
B = \frac{\sqrt{2}}{2\pi}\begin{pmatrix} \pi + i\ 6\ln 3 & -i\ 3 \\ 4\pi\ln 3 + i(12\ln^2 3 - \pi^2) & -\pi - i\ 6\ln 3 \end{pmatrix} \ . \tag{9.17}
$$

One can check that with the above values of the B_{ij} Eq.(9.16) holds also in the $z \to -1$ limit.

Substituting z by $-(z+1)/(z+9)$ Eq.(9.16) can also be written as

$$
\Psi_i(z - i\epsilon) = -\frac{2\sqrt{2}}{z+1}\sum_{j=1,2} B_{ij}\Psi_j\left(-\frac{z+9}{z+1} - i\epsilon\right) \ ; \tag{9.18}
$$

by chaining Eq.(9.16) and Eq.(9.18) one finds

$$
B \cdot B = -1 \ , \tag{9.19}
$$

satisfied of course by the explicit values Eq.(9.17), showing again that two of the four coefficients B_{ij} are fixed once the other two are given.

The transformation Eq.(9.13) maps $z = 3$ into $z = -3$. Correspondingly, from Eq.s(9.16,9.17) and Eq.(9.8) one obtains

$$
\begin{aligned}
Re\Psi_1(-3) &= \Psi_1(3) , \\
Im\Psi_1(-3) &= -\sqrt{3}\Psi_1(3) , \\
Re\Psi_2(-3) &= \frac{1}{3}(\sqrt{3}\pi + 6\ln 3)\Psi_1(3) , \\
Im\Psi_2(-3) &= (\pi - 2\sqrt{3}\ln 3)\Psi_1(3) .
\end{aligned}
\tag{9.20}
$$

In terms of the interpolating solutions, in the interval $0 < u < 1$, with $9 < (9 - u)/(1 - u) < \infty$, Eq.s(9.16) become

$$
\begin{aligned}
0 < \quad & u \quad < 1 \\
J_1^{(9,\infty)}\left(\frac{9-u}{1-u}\right) &= \frac{1-u}{2} J_1^{(0,1)}(u) , \\
J_2^{(9,\infty)}\left(\frac{9-u}{1-u}\right) &= \frac{1-u}{4} J_2^{(0,1)}(u) .
\end{aligned}
\tag{9.21}
$$

The range $1 < u < 9$ is mapped by the transformation into the (spacelike) interval $-\infty < -(9-u)/(u-1) < 0$, where we did not consider the interpolating solutions; the range $9 < u < \infty$, finally, is mapped into $0 < (u - 9)/(u - 1) < 1$, where Eq.s(9.16) give

$$
\begin{aligned}
9 < \quad & u \quad < \infty \\
J_1^{(0,1)}\left(\frac{u-9}{u-1}\right) &= \frac{u-1}{4} J_1^{(9,\infty)}(u) , \\
J_2^{(0,1)}\left(\frac{u-9}{u-1}\right) &= \frac{u-1}{2} J_2^{(9,\infty)}(u) .
\end{aligned}
\tag{9.22}
$$

The above relations are of course equivalent to Eq.s(9.21).

The transformations Eq.s(9.1,9.13) can be combined into a third transformation

$$
y = -9\ \frac{z+1}{z+9} , \qquad z = -9\ \frac{y+1}{y+9} , \tag{9.23}
$$

which maps the singular points of Eq.(4.3) into themselves, $0 \to -1 \to 0$, and $-9 \to \infty \to -9$. For real y, z it implies for the $i\epsilon$ prescription

$$
\begin{aligned}
-9\ \frac{z - i\epsilon + 1}{z - i\epsilon + 9} &= -9\ \frac{z+1}{z+9} + i\epsilon = y + i\epsilon , \\
-9\ \frac{y - i\epsilon + 1}{y - i\epsilon + 9} &= -9\ \frac{y+1}{y+9} + i\epsilon = z + i\epsilon .
\end{aligned}
\tag{9.24}
$$

Again, if $\Psi(z)$ is a solution of Eq.(4.3),

$$
\frac{1}{z+9}\Psi\left(-9\frac{z+1}{z+9}\right) \tag{9.25}
$$

is also a solution of the same equation, as it is easy to verify, and therefore can be expressed in terms of $\Psi_1(z)$ and $\Psi_2(z)$ as

$$
\Psi_i\left(-9\frac{z+1}{z+9} + i\epsilon\right) = \frac{\sqrt{2}}{12}(z+9)\sum_{j=1,2} C_{ij}\Psi_j(z - i\epsilon) , \tag{9.26}
$$

where C is a 2×2 matrix, and the overall constant $\sqrt{2}/12$ has been introduced for convenience.

By writing Eq.(9.6) with z replaced by $-9(z+1)/(z+9)$ and then using Eq,(9.16) one finds

$$\begin{aligned}\Psi_i\left(-9\frac{z+1}{z+9}+i\epsilon\right) &= -\frac{1}{3}\frac{z+9}{z+1}\sum_{j=1,2}A_{ij}\Psi_j\left(-\frac{z+9}{z+1}-i\epsilon\right)\\ &= -\frac{\sqrt{2}}{12}(z+9)\sum_{j=1,2}\sum_{k=1,2}A_{ij}B_{jk}\Psi_k(z-i\epsilon)\ . \qquad (9.27)\end{aligned}$$

By comparison with Eq.(9.26),

$$C=-A\cdot B=\frac{\sqrt{2}\sqrt{3}}{2\pi}\begin{pmatrix} 2\ln 3-i\pi & -1\\ 4\ln^2 3+\frac{1}{3}\pi^2 & -2\ln 3-i\,\pi\end{pmatrix}\ . \qquad (9.28)$$

Exchanging $(z-i\epsilon)$ with $-9(z+1)/(z+9)+i\epsilon$, Eq.(9.26 can also be written as

$$\Psi_i(z-i\epsilon)=\frac{6\sqrt{2}}{z+9}\sum_{j=1,2}C^*_{ij}\Psi_j\left(-9\frac{z+1}{z+9}+i\epsilon\right)\ . \qquad (9.29)$$

Note the appearance of C^*_{ij} in the r.h.s of the previous formula; indeed, the sign of $i\epsilon$ in the arguments has changed with respect to Eq.(9.26), and according to Eq.(8.3) that amounts to take the complex conjugate. By chaining Eq.(9.29) and Eq.(9.26) one gets the identity

$$C^*\cdot C=1\ , \qquad (9.30)$$

which is of course satisfied by the explicit values of the coefficients C_{ij} given by Eq.(9.28). In terms of the interpolating solutions, in the interval $0<u<1$, with $1>9(1-u)/(9-u)>0$, Eq.s(9.26) become

$$\begin{aligned} &0< u <1\\ J_1^{(0,1)}\left(9\frac{1-u}{9-u}\right) &= \frac{\sqrt{3}}{36}(9-u)J_2^{(0,1)}(u)\ ,\\ J_2^{(0,1)}\left(9\frac{1-u}{9-u}\right) &= \frac{\sqrt{3}}{6}(9-u)J_1^{(0,1)}(u)\ ; \qquad (9.31)\end{aligned}$$

the interval $1<u<9$ is mapped into the spacelike range $0>9(1-u)/(9-u)>-\infty$, where the interpolating functions were not considered, while for $9<u<\infty$, corresponding to $\infty>9(1-u)/(9-u)>9$, one finds the relations

$$\begin{aligned} &9< u <\infty\\ J_1^{(9,\infty)}\left(9\frac{u-1}{u-9}\right) &= \frac{\sqrt{3}}{18}(u-9)J_2^{(9,\infty)}(u)\ ;\\ J_2^{(9,\infty)}\left(9\frac{u-1}{u-9}\right) &= \frac{\sqrt{3}}{12}(u-9)J_1^{(9,\infty)}(u)\ , \qquad (9.32)\end{aligned}$$

which are of course equivalent to Eq.s(9.31).

10 The solution $S^{(0)}(2,z)$ at zeroth order in $(d-2)$.

We can now reconsider Eq.(4.5) for $n=0$, *i.e.* at zeroth order in the expansion in $(d-2)$. With the explicit value of $N^{(0)}(2,z)$ given by the first of Eq.s(4.2), Eq.(4.5) becomes

$$\begin{aligned} S^{(0)}(2,z) &= \Psi_1(z)\left(\Psi_1^{(0)}-\frac{1}{24}\int_0^z dw\ \Psi_2(w)\right)\\ &+ \Psi_2(z)\left(\Psi_2^{(0)}+\frac{1}{24}\int_0^z dw\ \Psi_1(w)\right)\ , \qquad (10.1)\end{aligned}$$

where the functions $\Psi_i(z)$ are known from the previous Sections, while the two integration constants are still to be fixed. To fix them, we will exploit the information that the solution, corresponding to the loop integral Eq.(refeq:defMI), is real as far as $u = -z$ is below the physical threshold $u = 9$, *i.e.* in the whole range $-\infty < u < 9$ or $\infty > z > -9$, with the reality condition valid for any d, in particular for the zeroth order term of the expansion around $d = 2$.

In the range $0 > z > -1$, *i.e.* $0 < u < 1$, using Eq.s(8.8) for expressing the $\Psi_i(z)$ in terms of the $J_i^{(0,1)}(u)$, the reality condition gives

$$Im\ S^{(0)}(2, z) = -\sqrt{3} J_2^{(0,1)}(u)\ \Psi_2^{(0)}\ , \tag{10.2}$$

which implies

$$\Psi_2^{(0)} = 0\ . \tag{10.3}$$

In the range $-1 > z > -9$, *i.e.* $1 < u < 9$, using Eq.s(8.8) and Eq.s(8.14) one obtains

$$Im\ S^{(0)}(2, z) = \frac{3}{\pi} \left(\sqrt{3}\Psi_1^{(0)} - \frac{1}{4} \int_0^1 dv\ J_1^{(0,1)}(v) \right) J_1^{(1,9)}(u)\ ; \tag{10.4}$$

the (qualitative) condition that $S^{(0)}(2, z)$ is real therefore gives the (quantitative) result

$$\Psi_1^{(0)} = \frac{\sqrt{3}}{12} \int_0^1 dv\ J_1^{(0,1)}(v)\ , \tag{10.5}$$

or, on account of Eq.(B.5) of the Appendix,

$$\Psi_1^{(0)} = \frac{\sqrt{3}}{12} \mathrm{Cl}_2\left(\frac{\pi}{3}\right), \tag{10.6}$$

where $\mathrm{Cl}_2(\phi)$ is the Clausen function of weight 2.

Eq.s(10.3,10.6) determine completely the solution Eq.(10.1), which from now on can be considered as known in closed analytic form. Indeed, expressing again, when needed, the $\Psi_i(z)$ in terms of the interpolating solutions $J_i(u)$, and on account of the results of the Appendix, we can evaluate in particular the behaviour of $S^{(0)}(2, z)$ at the singular points of Eq.(4.1) at $n = 0$. We find:

- at $z = 0$,

$$S^{(0)}(2, 0) = \Psi_1^{(0)} = \frac{\sqrt{3}}{12} \int_0^1 dv\ J_1^{(0,1)}(v) = \frac{\sqrt{3}}{12} \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ , \tag{10.7}$$

  in agreement with a result of [10];

- on the mass shell $-z = u = 1$

$$S^{(0)}(2, -1) = \frac{1}{16} \int_0^1 dv\ J_2^{(0,1)}(v) = \frac{1}{64}\pi^2\ ; \tag{10.8}$$

- at the threshold $(z + 9) \to 0^+$ or $-z = u \to 9^-$

$$S^{(0)}(2, z) \xrightarrow[(z+9)\to 0^+]{} -\frac{\sqrt{3}}{48}\left[\pi \ln\left(\frac{z+9}{72}\right) + 5\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right] + \mathcal{O}\big(z + 9\big)\ ; \tag{10.9}$$

- above threshold, with $z = -(u + i\epsilon)$ and $u > 9$, $S^{(0)}(2, z)$ develops the imaginary part

$$ImS^{(0)}(2, -u - i\epsilon) = \frac{1}{4}\pi\ J_2^{(9,\infty)}(u)\ , \tag{10.10}$$

  in agreement with the comments accompanying the introduction of Eq.(7.2) and Eq.(8.18);

- finally, when $z \to +\infty$ (spacelike region), the behaviour of $S^{(0)}(2,z)$ is

$$S^{(0)}(2,z) \xrightarrow[z\to+\infty]{} \frac{3}{16z} \ln^2 z + \mathcal{O}\left(\frac{1}{z^2}\right) . \tag{10.11}$$

The above results show explicitly that $S^{(0)}(2,z)$ is regular at $z=0$ and $z=-1$, as already recalled. It can be interesting to observe that by performing the change of variable $v \to 9/v$, and then using the first of Eq.s(9.11), Eq.(10.7) becomes

$$S^{(0)}(2,0) = \frac{1}{4}\int_9^\infty \frac{dv}{v} J_2^{(9,\infty)}(v) ; \tag{10.12}$$

similarly, with the change $v \to (v-9)/(v-1)$ and the second of Eq.s(9.22), Eq.(10.8) reads

$$S^{(0)}(2,-1) = \frac{1}{4}\int_9^\infty \frac{dv}{v-1} J_2^{(9,\infty)}(v) . \tag{10.13}$$

Quite in general, one can write for $S^{(0)}(2,z)$ the dispersion relation

$$S^{(0)}(2,z) = \frac{1}{\pi}\int_9^\infty \frac{dv}{v+z} Im S^{(0)}(2,-v-i\epsilon) = \frac{1}{4}\int_9^\infty \frac{dv}{v+z} J_2^{(9,\infty)}(v) , \tag{10.14}$$

where Eq.(10.10) has been used; Eq.(10.12) and Eq.(10.13) can then be seen as the dispersion relation evaluated at $z=0$ and $z=-1$.

We can also easily work out the expansions of $S^{(0)}(2,z)$ around any of the singular points points of Eq.(4.1) at $n=0$, namely $z=(0,-1,-9,\infty)$, with the above results providing the initial values and using either the known expansions of the $\Psi_i(z)$ given in Section 8 or, better, the differential equation Eq.(4.1) for obtaining the coefficients of the expansions up to any required order. We find:

- around $z=0$, according to Eq.(10.7), the expansion can be written as

$$S^{(0;0)}(2,z) = \Psi_1^{(0)} + \sum_{k=1,\infty} s_k^{(0)} z^k ; \tag{10.15}$$

  substituting in Eq.(4.1) with $n=0$, one finds

$$\begin{aligned} S^{(0;0)}(2,z) &= \Psi_1^{(0)}\psi_1^{(0)}(z) \\ &+ \frac{1}{24}z - \frac{23}{864}z^2 + ... , \end{aligned} \tag{10.16}$$

  where $\psi_1^{(0)}(z)$ is the same as in Eq.(6.2) ;

- around $z=-1$, according to Eq.(10.8), the expansion can be written as

$$S^{(0;1)}(2,z) = \frac{\pi^2}{64} + \sum_{k=1,\infty} s_k^{(1)}(z+1)^k , \tag{10.17}$$

  and the differential equation gives

$$\begin{aligned} S^{(0;1)}(2,z) &= \frac{\pi^2}{64}\psi_1^{(1)}(z) \\ &- \frac{3}{64}(z+1) - \frac{3}{128}(z+1)^2 + ... \end{aligned} \tag{10.18}$$

  where $\psi_1^{(1)}(z)$ is the same as in Eq.(6.7) ;

- around $z=-9$, according to Eq.(10.9), the expansion can be written as

$$S^{(0;9)}(2,z) = \ln(z+9) \sum_{k=0,\infty} s_k^{(9)}(z+9)^k + \sum_{k=0,\infty} t_k^{(9)}(z+9)^k \ , \tag{10.19}$$

with

$$s_0^{(9)} = -\frac{\sqrt{3}}{48}\pi \ , \qquad t_0^{(9)} = -\frac{\sqrt{3}}{48}\left[-\pi\ln(72) + 5\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right] \ ; \tag{10.20}$$

the differential equation gives

$$\begin{aligned} S^{(0;9)}(2,z) &= s_0^{(9)}\left[\ln(z+9)\ \psi_1^{(9)}(z) + \psi_2^{(9)}(z)\right] \\ &+ t_0^{(9)}\ \psi_1^{(9)}(z) \\ &+ \frac{1}{192}(z+9) + \frac{5}{6912}(z+9)^2 + ... \end{aligned} \tag{10.21}$$

where the $\psi_i^{(9)}(z)$ are the same as in Eq.(6.12) ;

- for large z, finally, according to Eq.(10.11) the expansion can be written as

$$S^{(0;\infty)}(2,z) = \frac{1}{z}\left[\ln^2 z \sum_{k=0,\infty} s_k^{(\infty)}\frac{1}{z^k} + \ln z \sum_{k=1,\infty} t_k^{(\infty)}\frac{1}{z^k} + \sum_{k=1,\infty} u_k^{(\infty)}\frac{1}{z^k}\right] \ , \tag{10.22}$$

with

$$s_0^{(\infty)} = \frac{3}{16} \ .$$

Imposing the validity of the differential equation, the expansion can be written as

$$\begin{aligned} S^{(0;\infty)}(2,z) &= \frac{3}{16z}\left[\ln^2 z\ \psi_1^{(\infty)}\left(\frac{1}{z}\right) - 2\ln z\ \psi_2^{(\infty)}\left(\frac{1}{z}\right)\right. \\ &+ \left.\frac{2}{z} - \frac{1}{2z^2} + ...\right] \ , \end{aligned} \tag{10.23}$$

where the $\psi_i^{(\infty)}(z)$ are the same as in Eq.(6.17).

The correspoding expansions for $dS^{(0)}(2,z)/dz$ is immediately obtained by differentiation. From Eq.(2.10) one can then obtain the z-expansions for $S_1^{(0)}(2,z)$, the zeroth order term in $(d-2)$ of the master integral $S_1(d,z)$ Eq.(2.5).

The above z-expansions can be used as the starting building blocks for implementing a computer routine for the fast and precise numerical evaluation of $S^{(0)}(2,z)$.

11 The solution $S^{(0)}(4,z)$ at zeroth order in $(d-4)$.

From the results of the previous Section and from Eq.(3.8) one immediately obtains the relevant z-expansions for $S^{(0)}(4,z)$, the zeroth order term in $(d-4)$ of $S(d,z)$ Eq.(2.4):

- around $z=0$

$$\begin{aligned} S^{(0;0)}(4,z) &= \Psi_1^{(0)}\left(\frac{3}{2} + \frac{1}{3}z - \frac{1}{27}z^2 + ...\right) \\ &- \frac{21}{32} - \frac{3}{128}z + \frac{11}{1728}z^2 + ... \ , \end{aligned} \tag{11.1}$$

with $\Psi_1^{(0)}$ given by Eq.(10.6);

- around $z = -1$

$$
\begin{aligned}
S^{(0;1)}(4,z) &= \frac{\pi^2}{64}\left(-\frac{1}{2}(z+1)^2 - \frac{5}{8}(z+1)^3 + ...\right) \\
&- \frac{59}{128} + \frac{3}{128}(z+1) + \frac{5}{64}(z+1)^2 + \frac{37}{384}(z+1)^3 + ... ; \qquad (11.2)
\end{aligned}
$$

- around $z = -9$

$$
\begin{aligned}
S^{(0;9)}(4,z) &= \left(s_0^{(9)}\ln(z+9) + t_0^{(9)}\right)\left(\frac{1}{54}(z+9)^2 + \frac{1}{648}(z+9)^3 + ...\right) \\
&+ s_0^{(9)}\left(8 - \frac{4}{9}(z+9) - \frac{4}{9}(z+9)^2 + \frac{7}{5832}(z+9)^3 + ...\right) \\
&+ \frac{45}{128} - \frac{23}{384}(z+9) - \frac{1}{1728}(z+9)^2 + \frac{1}{10368}(z+9)^3 + ... , \qquad (11.3)
\end{aligned}
$$

with $s_0^{(9)}$ and $t_0^{(9)}$ as in Eq.(10.20);

- for large and positive z (spacelike u), finally

$$
\begin{aligned}
S^{(0;\infty)}(4,z) &= \ln^2 z\left(\frac{3}{32} + \frac{3}{16z} - \frac{3}{16z^2} + ...\right) \\
&+ \ln z\left(\frac{1}{32}z + \frac{9}{32z} + \frac{3}{8z^2} + ...\right) \\
&- \frac{13}{128}z - \frac{15}{32} - \frac{3}{64z} + \frac{29}{64z^2} + ... \qquad (11.4)
\end{aligned}
$$

The above expansions can be used as the starting building blocks for the fast and precise numerical evaluation of $S^{(0)}(4,z)$.

12 The solution $S^{(1)}(2,z)$ at first order in $(d-2)$.

Once $S^{(0)}(2,z)$, is known, $N^{(1)}(2,z)$, given by the second of Eq.s(4.2), is also known, so that we can work out Eq.(4.1) and its solution Eq.(4.5) for $n = 1$, *i.e.* at first order in the expansion in $(d-2)$. After some algebra, it takes the relatively simple expression

$$
\begin{aligned}
S^{(1)}(2,z) &= \Psi_1(z)\Bigg[\Psi_1^{(1)} + \Psi_1^{(0)}\left(-\ln 3 - \frac{1}{4}\ln z + \frac{1}{2}\ln(z+1) + \frac{1}{2}\ln(z+9)\right) \\
&\qquad - \frac{1}{96}\int_0^z dw\left(4 - \frac{1}{w} + \frac{2}{w+1} + \frac{2}{w+9}\right)\Psi_2(w)\Bigg] \\
&+ \Psi_2(z)\Bigg[\Psi_2^{(1)} + \frac{1}{4}\Psi_1^{(0)} \\
&\qquad + \frac{1}{96}\int_0^z dw\left(4 - \frac{1}{w} + \frac{2}{w+1} + \frac{2}{w+9}\right)\Psi_1(w)\Bigg] , \qquad (12.1)
\end{aligned}
$$

where $\Psi_1^{(0)}$ is given by Eq.(10.6) (according to Eq.(10.3) $\Psi_2^{(0)} = 0$) and the two integration constants $\Psi_i^{(1)}$ are still to be determined. Imposing as in the previous Section the reality conditions for $\infty > z > -9$, we find

$$
\Psi_2^{(1)} = 0 , \qquad (12.2)
$$

and

$$\Psi_1^{(1)} = \frac{\sqrt{3}}{24}\int_0^1 dv\left[\frac{\pi}{3}J_2^{(0,1)}(v)+\left(2+\ln 3+\frac{1}{2}\ln v-\ln(1-v)-\ln(9-v)\right)J_1^{(0,1)}(v)\right]$$
$$= \frac{\sqrt{3}}{12}\left[\mathrm{Cl}_2\left(\frac{\pi}{3}\right)-\frac{3}{4}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{1}{2}\ln 3\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)-\frac{1}{24}\pi^3\right] , \qquad (12.3)$$

where use is made of the definite integrals of the Appendix.

$S^{(1)}(2,z)$ is now fully determined in closed analytic form. Expressing again the $\Psi_i(z)$ in terms of the interpolating solutions $J_i(u)$ and on account of the results of the Appendix, we obtain in particular the following behaviours:

- at $z=0$

$$S^{(1)}(2,0)=\Psi_1^{(1)}=\frac{\sqrt{3}}{12}\left[\mathrm{Cl}_2\left(\frac{\pi}{3}\right)-\frac{3}{4}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{1}{2}\ln 3\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)-\frac{1}{24}\pi^3\right] , \qquad (12.4)$$

  in agreement with [10] if allowance is made for the difference in our integration measure Eq.(2.1), which generates the extra term $\mathrm{Cl}(\pi/3)$;

- on the mass shell $-z=u=1$

$$S^{(1)}(2,-1)=\frac{1}{64}\left(\pi^2+3\pi^2\ln 2-\frac{21}{2}\zeta(3)\right) ; \qquad (12.5)$$

- at the threshold $(z+9)\to 0^+$ or $-z=u\to 9^-$

$$S^{(1)}(2,z) \xrightarrow[(z+9)\to 0^+]{} -\sqrt{3}\frac{\pi}{96}\ln\left(\frac{z+9}{72}\right)\left[\ln(z+9)+3\ln 2-\ln 3+2\right]$$
$$+\sqrt{3}\frac{5}{64}\left[\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)-\frac{2}{3}(2+6\ln 2+\ln 3)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right]$$
$$+\sqrt{3}\frac{1}{48}\beta_3+\sqrt{3}\frac{17}{3456}\pi^3+\mathcal{O}\big(z+9\big) ; \qquad (12.6)$$

- finally, when $z\to+\infty$ (spacelike region), the behaviour of $S^{(1)}(2,z)$ is

$$S^{(1)}(2,z) \xrightarrow[z\to+\infty]{} \frac{1}{32z}\left[3\ln^3 z+6\ln^2 z-\pi^2\ln z+18\zeta(3)\right]+\mathcal{O}\left(\frac{1}{z^2}\right) . \qquad (12.7)$$

The corresponding complete expansions in z for $S^{(1)}(2,z)$, $dS^{(1)}(2,z)/dz$, $S^{(1)}(4,z)$ etc. can be obtained as in the case of $S^{(0)}(2,z)$; we omit them for the sake of brevity. Let us just observe that from Eq.(3.9) and Eq.(12.5) one gets

$$S^{(1)}(4,-1) = \frac{8}{3}S^{(0)}(2,-1)+\frac{65}{512} = \frac{\pi^2}{24}+\frac{65}{512} , labbel S(1,4,-1) \qquad (12.8)$$

in agreement with the known result [11].

13 Conclusions.

We have considered the system of first order differential equations in the external momentum transfer satisfied by the Master Integrals of the two loop sunrise graph in the equal mass limit and in the usual continuous d-dimensional regularisation. The system is equivalent to a single second order equation for the scalar MI; after recalling the relation between the expansions in $(d-4)$ and $(d-2)$, the second order

equation is expanded around $d = 2$ and solved by means of the variation of the constants formula by Euler. That requires the knowledge of the solutions of the homogeneous equation, which is obtained by working out first the series expansions of the differential equation at all its singular points, then joining smoothly those expansions by means of a set of "interpolating solutions", whose explicit form (one-dimensional definite integrals of a suitable integrand) is suggested by the Cutkosky-Veltman rule for the imaginary part of the MI. The interpolating solutions, whose relation to the complete elliptic integrals of the first kind is discussed in an Appendix, are found to transform on themselves for a particular set of transformations of the arguments. Once the full knowledge of the homogeneous equation is established, Euler's formula provides in closed analytic form also the limiting values of the solutions at all the singular points of the equation. They are worked out explicitly for the zeroth order in $(d-2)$, the related zeroth order in $(d-4)$ and the first order in $(d-2)$.

The analytic behaviour at the singular points and the related expansions can be used as the starting building blocks for the implementation of a fast and precise numerical routine for the evaluation of the equal-mass sunrise scalar integral for arbitrary values of the momentum transfer. The extension of the approach to other two-loop self-mass amplitudes with a single mass scale seems easy to obtain; in that way one could complete the analytic and numerical evaluation of the two-loop electron self-mass in QED. While such a result is somewhat academical, much more interesting, even if correspondingly difficult (but not out of reach) will be the extension of the approach to the two-loop self-mass integrals in the general mass case, as required by the precision tests of the current Standard Model (QCD & EW interactions) of elementary particles.

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A Relations with the Complete Elliptic Integrals.

The integral Eq.(7.2), which is also the phase space of three particles of equal mass, can be written as a Complete Elliptic Integral of the First Kind [9], (the first explicit reference to phase space integrals and elliptic integrals was perhaps made in [13]); the same is true of course for all the other integrals corresponding to the interpolating solutions.

Quite in general, given a polynomial of fourth order in b, written as

$$R_4(b) = (b - b_1)(b - b_2)(b - b_3)(b - b_4) \ , \tag{A.1}$$

where the four roots are ordered as

$$b_1 < b_2 < b_3 < b_4 \ , \tag{A.2}$$

and the integral

$$I(b_1, b_2) = \int_{b_1}^{b_2} \frac{db}{\sqrt{-R_4(b)}} \ , \tag{A.3}$$

performing the standard change of variable [14]

$$x^2 = \frac{(b_4 - b_2)(b - b_1)}{(b_2 - b_1)(b_4 - b)} \ , \qquad b = \frac{b_4(b_2 - b_1)x^2 + b_1(b_4 - b_2)}{(b_1 - b_1)x^2 + (b_4 - b_2)} \ , \tag{A.4}$$

one obtains

$$I(b_1, b_2) = \int_{b_1}^{b_2} \frac{db}{\sqrt{-R_4(b)}} = \frac{2}{\sqrt{(b_4 - b_2)(b_3 - b_1)}} K(m) \ , \tag{A.5}$$

where

$$m = \frac{(b_2 - b_1)(b_4 - b_3)}{(b_4 - b_2)(b_3 - b_1)} \tag{A.6}$$

and $K(m)$ is the Complete Elliptic Integral of the First Kind,

$$K(m) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-mx^2)}} \ . \tag{A.7}$$

Similarly, one finds

$$I(b_2, b_3) = \int_{b_2}^{b_3} \frac{db}{\sqrt{R_4(b)}} = \frac{2}{\sqrt{(b_4 - b_2)(b_3 - b_1)}} K(1-m) \ , \tag{A.8}$$

where m is the same as in Eq.(A.6), and finally

$$I(b_3, b_4) = \int_{b_3}^{b_4} \frac{db}{\sqrt{-R_4(b)}} = \frac{2}{\sqrt{(b_4 - b_2)(b_3 - b_1)}} K(m) \ , \tag{A.9}$$

so that, quite in general

$$I(b_1, b_2) = I(b_3, b_4) \ , \tag{A.10}$$

in agreement with Eq.s(7.11,7.13).

By using the above results, we can express the interpolating solutions introduced in Section 8 in terms of complete elliptic integrals. We find:

- for the interpolating solutions in the interval $0 < u < 1$, defined by Eq.s(8.5),

$$\begin{aligned} J_1^{(0,1)}(u) &= \frac{1}{\sqrt{(1+\sqrt{u})^3(3-\sqrt{u})}} K\big(a(u)\big) \ , \\ J_2^{(0,1)}(u) &= \frac{1}{\sqrt{(1+\sqrt{u})^3(3-\sqrt{u})}} K\big(1-a(u)\big) \ , \\ a(u) &= \frac{(1-\sqrt{u})^3(3+\sqrt{u})}{(1+\sqrt{u})^3(3-\sqrt{u})} \ ; \end{aligned} \tag{A.11}$$

- for the interpolating solutions in the interval $1 < u < 9$, defined by Eq.s(8.12),

$$\begin{aligned} J_1^{(1,9)}(u) &= \frac{1}{\sqrt{16\sqrt{u}}} K\big(b(u)\big) \ , \\ J_2^{(1,9)}(u) &= \frac{1}{\sqrt{16\sqrt{u}}} K\big(1-b(u)\big) \ , \\ b(u) &= \frac{(\sqrt{u}-1)^3(3+\sqrt{u})}{16\sqrt{u}} \ ; \end{aligned} \tag{A.12}$$

- finally, for the interpolating solutions in the interval $9 < u < \infty$, defined by Eq.s(8.18),

$$\begin{aligned} J_1^{(9,\infty)}(u) &= \frac{1}{\sqrt{(\sqrt{u}-1)^{(}\sqrt{u}+3)}} K\big(c(u)\big) \ , \\ J_2^{(9,\infty)}(u) &= \frac{1}{\sqrt{(\sqrt{u}-1)^{(}\sqrt{u}+3)}} K\big(1-c(u)\big) \ , \\ c(u) &= \frac{16\sqrt{u}}{(\sqrt{u}-1)^3(3+\sqrt{u})} \ . \end{aligned} \tag{A.13}$$

The identities under transformations of the argument, written in Section 9 for the interpolating solutions, can be rewritten in terms of the elliptic integrals. We obtain, in terms of the arguments $a(u), b(u), c(u)$ defined in Eq.s(A.11,A.12,A.13):

- from Eq.s(9.9) or Eq.s(9.11)
$$\frac{K\big(1-a(u)\big)}{K\big(a(u)\big)} = 3\frac{K\big(c(t)\big)}{K\big(1-c(t)\big)} \ , \tag{A.14}$$
with $t = 9/u$;
- from Eq.s(9.10)
$$\frac{K\big(1-b(u)\big)}{K\big(b(u)\big)} = 3\frac{K\big(b(t)\big)}{K\big(1-b(t)\big)} \ , \tag{A.15}$$
again with $t = 9/u$;
- from Eq.s(9.22) or Eq.s(9.21)
$$\frac{K\big(1-a(u)\big)}{K\big(a(u)\big)} = 2\frac{K\big(1-c(v)\big)}{K\big(c(v)\big)} \ , \tag{A.16}$$
with $v = (u-9)/(u-1)$;
- from Eq.s(9.31)
$$\frac{K\big(1-a(u)\big)}{K\big(a(u)\big)} = 6\frac{K\big(a(w)\big)}{K\big(1-a(w)\big)} \ , \tag{A.17}$$
with $w = 9(u-1)/(u-9)$;
- finally from Eq.s(9.32)
$$\frac{K\big(1-c(u)\big)}{K\big(c(u)\big)} = \frac{3}{2}\frac{K\big(c(w)\big)}{K\big(1-c(w)\big)} \ . \tag{A.18}$$
again with $w = 9(u-1)/(u-9)$.

Eq.(A.14) is based on the algebraic transformation $a(u) \to 1-c(9/u)$; when the parameter u is in the range $0 < u < 1$ all the arguments are real and in the range $(0,1)$, and all the elliptic functions take real and positive values (the equation remains of course valid, by analytic continuation, for any value of u). Similarly, all the arguments are in the range $(0,1)$ in Eq.s(A.16) and (A.17) also for $0 < u < 1$, in Eq.(A.15) for $1 < u < 9$ and for $9 < u < \infty$ in Eq.(A.18), but the relations can be continued to arbitrary values of u. The algebraic relation derived from Eq.(A.14) by eliminating the parameter u between $a(u)$ and $1-c(9/u)$ is a modular equation of degree 3 [15]; similarly, Eq.(A.15) gives a modular equation also of degree 3 between $b(u)$ and $1-b(9/u)$, Eq.(A.16) a modular equation of degree 2 between $a(u)$ and $c((u-9)/(u-1))$, Eq.(A.17) an equation of degree 6 between $a(u)$ and $1-a(9(1-u)/(9-u))$ and finally Eq.(A.18) a modular equation of degree 3/2 between $c(u)$ and $1-c(9(1-u)/(9-u))$.

B Definite Integrals.

In this appendix we discuss the evaluation of the definite integrals used in the previous Sections. As a first case, we consider
$$\int_0^1 du\ J_1^{(0,1)}(u) = \int_0^1 du \int_0^{(1-\sqrt{u})^2} \frac{db}{\sqrt{-R_4(u,b)}} \ , \tag{B.1}$$

where $J_1^{(0,1)}(u)$ is defined by Eq.(8.5) and $R_4(u,b)$ by Eq.(7.3). One can invert the order of integration, obtaining

$$\begin{aligned}\int_0^1 du\ J_1^{(0,1)}(u) &= \int_0^1 \frac{db}{\sqrt{b(4-b)}} \int_0^{(1-\sqrt{b})^2} \frac{du}{\sqrt{((1-\sqrt{b})^2-u)((1+\sqrt{b})^2-u)}} \\ &= -\frac{1}{2}\int_0^1 \frac{db}{\sqrt{b(4-b)}} \ln b\ , \end{aligned} \tag{B.2}$$

where the u-integration is trivial. One can then perform the standard change of variable

$$b = \frac{(1+t)^2}{t}\ , \tag{B.3}$$

which for $0 < b < 4$ is inverted as

$$t = \frac{\sqrt{b} - i\sqrt{4-b}}{\sqrt{b} + i\sqrt{4-b}}\ , \tag{B.4}$$

with t varying in the unit circle between $t = -1$ at $b = 0$ and $t = \mathrm{e}^{-2i\pi/3}$ at $b = 1$; one finds

$$\begin{aligned}\int_0^1 du\ J_1^{(0,1)}(u) &= \frac{i}{2}\int_{-1}^{\mathrm{e}^{-2i\pi/3}} \frac{dt}{t}\left(\ 2\ln(1+t) - \ln t\ \right) \\ &= i\left[-\mathrm{Li}_2(-t) - \frac{1}{4}\ln^2 t\right]_{-1}^{\mathrm{e}^{-2i\pi/3}} \\ &= \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ , \end{aligned} \tag{B.5}$$

where use has been made of the formula

$$\mathrm{Li}_2(\mathrm{e}^{i\phi}) = \frac{1}{6}\pi^2 - \frac{1}{2}\pi\phi + \frac{1}{4}\phi^2\ + i\mathrm{Cl}_2(\phi),$$

and

$$\mathrm{Cl}_2(\phi) = -\int_0^{\phi} d\theta \ln\left(2\sin\frac{\theta}{2}\right) \tag{B.6}$$

is the Clausen function of weight 2.

Along the same lines, one easily obtains

$$\begin{aligned}\int_0^1 du\ J_2^{(0,1)}(u) &= \int_0^4 \frac{db}{\sqrt{b(4-b)}} \int_{(1-\sqrt{b})^2}^{1} \frac{du}{\sqrt{((1+\sqrt{b})^2-u)(u-(1-\sqrt{b})^2)}} \\ &= \frac{i}{2}\int_0^4 \frac{db}{\sqrt{b(4-b)}} \ln\frac{\sqrt{b}-i\sqrt{4-b}}{\sqrt{b}+i\sqrt{4-b}} \\ &= \frac{1}{4}\pi^2\ , \end{aligned} \tag{B.7}$$

and, as expected from Eq.(8.6),

$$\int_0^1 du\ J_3^{(0,1)}(u) = \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ .$$

The same approach gives also at once

$$\int_1^9 du\ J_2^{(1,9)}(u) = \frac{3}{4}\pi^2\ . \tag{B.8}$$

The case of $J_1^{(1,9)}(u)$ is more complicated; indeed, following the same procedure as for the previous integrals one finds

$$\begin{aligned}\int_1^9 du\ J_1^{(1,9)}(u) &= \int_0^4 \frac{db}{\sqrt{b(4-b)}} \int_{(\sqrt{b}+1)^2}^9 \frac{du}{\sqrt{(u-(1+\sqrt{b})^2)(u-(1-\sqrt{b})^2)}} \\ &= \frac{1}{2}\int_0^4 \frac{db}{\sqrt{b(4-b)}} \ln\frac{8-b+\sqrt{(4-b)(16-b)}}{8-b-\sqrt{(4-b)(16-b)}}\ ,\end{aligned} \tag{B.9}$$

where the new square root $\sqrt{16-b}$ has appeared. If one uses at this point the change of variable Eq.s(B.3), as a consequence of the factor $\sqrt{16-b}$ in the argument of the logarithm a new quadratic square root, $\sqrt{1-14t+t^2}$, appears; that square root can be in turn removed by a suitable change of variable, and the result can be expressed, by brute force, as a combination of many dilogarithms of unusual and rather nasty arguments. It can be more convenient to consider the auxiliary integral

$$A(m) = \int_0^4 \frac{db}{\sqrt{b(4-b)}} \int_{(\sqrt{b}+m)^2}^{(2+m)^2} \frac{du}{\sqrt{(u-(m+\sqrt{b})^2)(u-(m-\sqrt{b})^2)}}\ . \tag{B.10}$$

Taking the derivative with respect of m^2 of the integral representation for $A(m)$ gives

$$\frac{d}{dm^2}A(m) = -\frac{i}{2m^2}\int_0^4 \frac{db}{\sqrt{b(b-4(m+1)^2)}} = -\frac{1}{2m^2}(-2\ln x - i\pi)\ ,$$

where x is defined through $m+1 = 2x/(x^2+1)$. The primitive in x of that expression is

$$\begin{aligned}A(m(x)) &= i\Big[2i\pi\ln(1-x) - i\pi\ln(1+x^2) + 4\ln x\ln(1-x) + 4\mathrm{Li}_2(x) \\ &\quad - \ln(x^2)\ln(1+x^2) - \mathrm{Li}_2(-x^2) + \frac{1}{4}\pi^2\Big]\ ,\end{aligned} \tag{B.11}$$

where the constant term $\pi^2/4$ is fixed by observing that $A(m=-1)$ is equal to the integral Eq.(B.7) times i. At $m=1$, or $x = \mathrm{e}^{-i\pi/3}$, Eq.(B.11) gives

$$\int_1^9 du\ J_1^{(1,9)}(u) = 5\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ . \tag{B.12}$$

The same result can be obtained by using the identity Eq.(9.10); by the change of variable $u = 9/v$ one has

$$\int_1^9 du\ J_1^{(1,9)}(u) = \frac{\sqrt{3}}{9}\int_1^9 dv\ \frac{1}{v}J_2^{(1,9)}(v)\ , \tag{B.13}$$

and the integration can be continued, as for Eq.(B.8), along the line followed for Eq.s(B.2,B.7).

With a suitable combinations of the above described techniques, one obtains also, with $U >> 1$

$$\begin{aligned}
\int_9^U du\ J_1^{(9,\infty)}(u) &= \pi \ln U - 5\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \\
\int_9^U du\ J_2^{(9,\infty)}(u) &= \frac{3}{4}\ln^2 U - \frac{1}{4}\pi^2 \ .
\end{aligned} \tag{B.14}$$

The extension of the above formulae to the case in which a logarithmic factor is also present in the integrand requires much additional effort. We list the results

$$\begin{aligned}
\int_0^1 du\ J_1^{(0,1)}(u) \ln u &= 3\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{1}{6}\pi^3 \ , \\
\int_0^1 du\ J_2^{(0,1)}(u) \ln u &= 2\pi\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - 7\zeta(3) \ , \\
\int_0^1 du\ J_1^{(0,1)}(u) \ln(1-u) &= -\frac{1}{36}\pi^3 \ , \\
\int_0^1 du\ J_2^{(0,1)}(u) \ln(1-u) &= -\frac{21}{8}\zeta(3) \ , \\
\int_0^1 du\ J_1^{(0,1)}(u) \ln(9-u) &= 3\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{5}{18}\pi^3 \ , \\
\int_0^1 du\ J_2^{(0,1)}(u) \ln(9-u) &= \frac{35}{8}\zeta(3) \ ,
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
\int_1^9 du\ J_1^{(1,9)}(u) \ln u &= 15\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{23}{18}\pi^3 \ , \\
\int_1^9 du\ J_2^{(1,9)}(u) \ln u &= 7\zeta(3) \ , \\
\int_1^9 du\ J_1^{(1,9)}(u) \ln(u-1) &= \beta_3 \ , \\
\int_1^9 du\ J_2^{(1,9)}(u) \ln(u-1) &= \frac{21}{8}\zeta(3) \ , \\
\int_1^9 du\ J_1^{(1,9)}(u) \ln(9-u) &= 15\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{37}{36}\pi^3 + \beta_3 \ , \\
\int_1^9 du\ J_2^{(1,9)}(u) \ln(9-u) &= 5\pi\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{35}{8}\zeta(3) \ ,
\end{aligned} \tag{B.16}$$

$$
\begin{aligned}
\int_9^U du\ J_1^{(9,\infty)}(u)\ln u &= \frac{1}{2}\pi\ln^2 U - \frac{10}{9}\pi^3 - 15\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)\ , \\
\int_9^U du\ J_2^{(9,\infty)}(u)\ln u &= \frac{1}{2}\ln^3 U - \zeta(3)\ , \\
\int_9^U du\ J_1^{(9,\infty)}(u)\ln(u-1) &= \frac{1}{2}\pi\ln^2 U + \frac{1}{12}\pi^3 - \beta_3\ , \\
\int_9^U du\ J_2^{(9,\infty)}(u)\ln(u-1) &= \frac{1}{2}\ln^3 U - \frac{3}{2}\zeta(3)\ , \\
\int_9^U du\ J_1^{(9,\infty)}(u)\ln(u-9) &= \frac{1}{2}\pi\ln^2 U - \frac{43}{36}\pi^3 - 15\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) - \beta_3\ , \\
\int_9^U du\ J_2^{(9,\infty)}(u)\ln(u-9) &= \frac{1}{2}\ln^3 U + \frac{11}{2}\zeta(3) - 5\pi\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ . \qquad \text{(B.17)}
\end{aligned}
$$

The constants $\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)$ and β_3 appearing in the above equations are defined as

$$
\begin{aligned}
\mathrm{Ls}_3(\phi) &= -\int_0^\phi d\theta\ln^2\left(2\sin\frac{\theta}{2}\right)\ , \\
\beta_3 &= -\frac{3}{4}\pi\ln^2 3 - \frac{3}{2}\pi\mathrm{Li}_2\left(-\frac{1}{3}\right) - 18\ Im\mathrm{S}_{12}(i\sqrt{3})\ . \qquad \text{(B.18)}
\end{aligned}
$$

As a final remark, in obtaining the previous results we obtained also the relations

$$
\begin{aligned}
\mathrm{Cl}_2\left(\frac{\pi}{6}\right) &= \frac{2}{3}G + \frac{1}{4}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ , \\
\mathrm{Ls}_3(\phi) + \mathrm{Ls}_3(\pi-\phi) &= -\frac{1}{6}\pi^3\ , \qquad \text{(B.19)}
\end{aligned}
$$

where G is the Catalan constant, which we could not find in the literature.

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